LVIII. *A treatiſe on the preceſſion of the equinoxes, and in general on the motion of the nodes, and the alteration of the inclination of the orbit of a planet to the ecliptic. Inſcribed to the gentlemen of the* Royal Society, *by M.* De St. Jaques Silvabelle.

Tranſlated from the French *M. S. by* J. Bevis, *M. D.*

Introduction.

Read March 12, 1752.

IF the earth were perfectly ſpherical, the action of the ſun on all the parts which compoſe it, would not produce any effect to make it turn round its centre; becauſe the moment, which would be produced on one ſide, would be always counterbalanced by an equal moment on the oppoſite ſide of the centre.

It would be the ſame, if the earth were a ſpheroid flatted at the poles, and the ſun was always in the equator, or in the ninetieth degree of declination: But in every other degree of declination its action on the exceſs of matter about the equator has a tendency to make the equator approach towards the ſun's place, or to diminiſh the angle of the ſun's declination, by making the earth's axis to turn round its centre in the plane of the circle of the ſun's declination.

The earth has then, at every inſtant, two motions of rotation; one about the axe of the equator, called alſo the earth's axe; and this is the diurnal motion,

 which

which is uniform; the other motion of rotation is performed about the axe of the circle of the sun's declination, which is a diameter of the equator; and this motion is produced by the action of the sun on the redundant matter about the equator, and is continually accelerated, from the continual application of the solar action producing it.

The point E, *fig.* 1. *n*° 2. which is the intersection of the circumference of the equator and the circumference of the circle of declination, has two motions, whose directions are perpendicular to each other. Let $E\,e$ be the space, which it runs through in an instant $d\,t$, in the circumference of the equator, by the uniform diurnal motion, and let $E\,\varepsilon$ be the space it runs through in the same instant, in the circumference of the circle of declination, by an accelerated motion, as has been explained.

The point E, in vertue of these two motions $E\,e$ and $E\,\varepsilon$, will not circulate either in the circumference $E\,e\,Q\,e'\,Q\,E$ of the equator, or in the circumference $E\,\varepsilon\,P\,E'\,P'\,E$ of the circle of declination, but forming the rectangular parallelogram $E\,e\,e'\,\varepsilon$, the diagonal $E\,e'$ will be the elementary arc of the circumference $E\,e\,q\,E'$, in which the point E will circulate, and the angle $e\,E\,e'$ will be equal to the angle $Q\,C\,q$, and equal to the angle $P'\,C\,p'$ which the pole P' runs through in an instant in the circumference $P'\,p'\,Q\,q\,P\,Q'\,P'$ whose plane is perpendicular to the plane of the circle of declination; and when the lines $E\,e$, $E\,\varepsilon$ are known at every instant, $P'\,p'$ will also be known, since the angle $e\,E\,e'$ = to the angle $Q\,C\,q$ = to the angle $P'\,C\,p'$.

The

Fig. 1. N° 1.
p. 392.

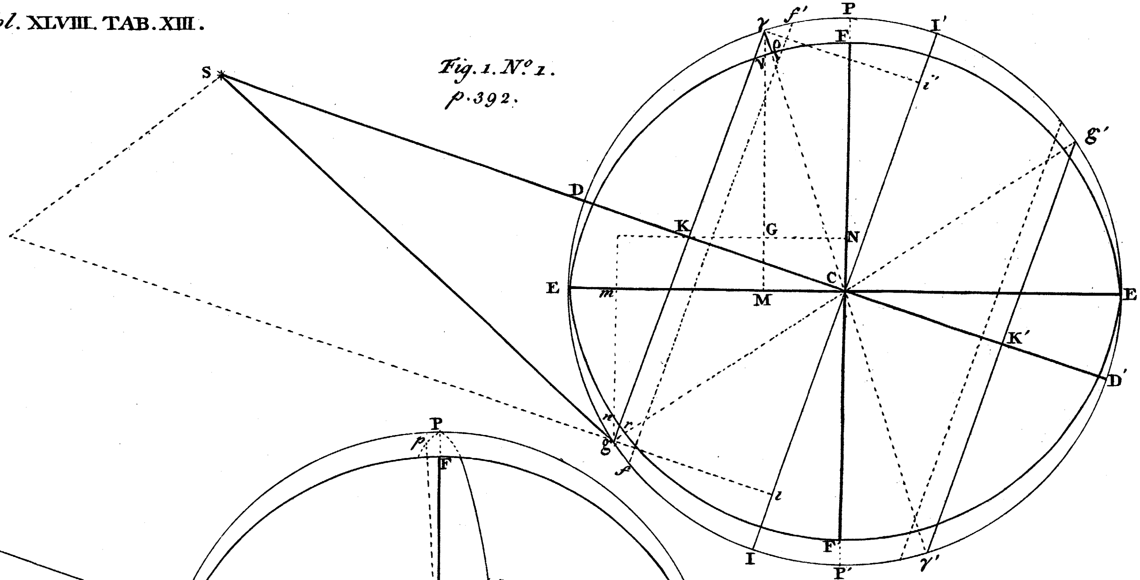
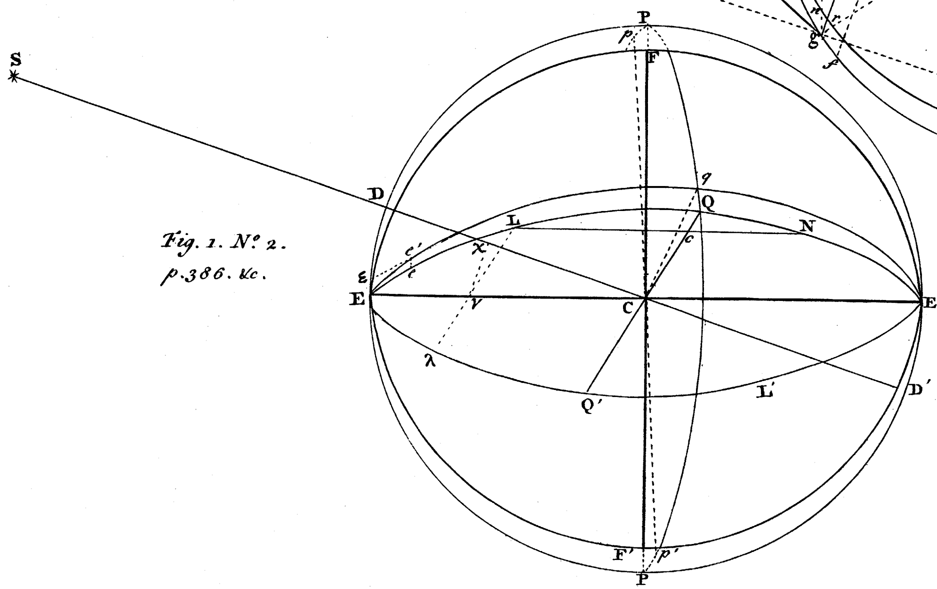


Fig. 1. N° 2.
p. 386. &c.



The inſtantaneous motion of the pole, which is $P'p'$, or Pp, *fig.* 2. *n*° 1. & *n*° 2*. may be reſolved into two, $P\,R$ and $P\,M$, perpendicular to each other, and both to the earth's axe. The former cauſes the pole P to move parallel to the ecliptic ♈ ♋ ♎ ♑, and alters the place of the ſolſtice ♋, and conſequently alſo that of the equinoctial points ♈ and ♎; the latter, which is according to $P\,M$, alters the inclination of the earth's axe to the ecliptic.

To have the motion of the pole parallel to the ecliptic, or, which is the ſame, the motion of the node ♈, or the preceſſion, in the ſame time that the ſun paſſes from the equinox ♈ to the ſolſtice ♋, take the integral of the lines $P\,R$, ſuppoſing $P\,R$ generally to expreſs the inſtantaneous preceſſion for any declination of the ſun S.

And to have the alteration of the inclination in the ſame time that the ſun is paſſing from ♈ to ♋, take the ſum, or the integral of the lines $P\,M$, ſuppoſing $P\,M$ generally to expreſs the inſtantaneous alteration of the inclination of the earth's axe to the ecliptic for any given declination of the ſun.

The ſum of the lines $P\,R$ is always the ſame, and has the ſame ſign, or the ſame direction, during every quarter of the ſun's revolution, whether he moves from ♈ to ♋, or from ♋ to ♎, or from ♎ to ♑, or from ♑ to ♈; ſo that the preceſſion anſwering to any one quarter of the ſun's revolution about the earth, or to three months, being known, that multiplied by 4 will be the annual preceſſion; by 8 will give it for two years; by 16 for 4 years, &c.

 Likewiſe

* *Fig.* 2. is explained at the beginning of *Prob.* 5.

Likewiſe the ſum of the lines *P M* is ever the ſame for every quarter of the ſolar revolution; but it has alternatively a contrary ſign; that is, a contrary direction. During the quarter from ♈ to ♋, the alteration of the inclination of the earth's axe to the ecliptic is poſitive, and the angle of the inclination increaſes; but during the ſucceeding quarter, or from ♋ to ♎, the alteration of the inclination is negative, and the angle of the inclination diminiſhes: And as the diminution from ♋ to ♎ is equal to the augmentation from ♈ to ♋, it follows, that at the end of the ſemirevolution the inclination of the earth's axe to the plane of the ecliptic will become again the ſame, having undergone an oſcillation, which is completed in a ſemirevolution. It is the ſame, when the ſun paſſes from ♎ to ♈. The angle of the inclination increaſes from ♎ to ♑, and decreaſes from ♑ to ♈, where it becomes again the ſame it was at ♎.

And hence the inclination of the earth's axe to the ecliptic may be conſidered as conſtant, tho' ſubject to this oſcillation, and indeed to ſeveral others, which will be preſently explained, they being all regular, and performed in regular periods.

The earth's inclination to the ecliptic being conſtant, and the motion of the pole which produces the preceſſion, being always parallel to the plane of the ecliptic, the earth's pole moves in a parallel to the ecliptic, about 23 degrees and a half diſtant from the pole of the ecliptic, and the terreſtrial axe deſcribes a conic ſurface.

To this motion of the terreſtrial axe or pole is to be aſcribed the apparent motion of the ſtars about the pole of the ecliptic.

But

But hitherto we have not confidered, that to the preceffion, thus caufed by the fun, we are to add that likewife produced by the moon; and it remains, that we examine into the motion of the earth's pole, caufed by the action of the moon on the redundant matter about the earth's equator.

All, that has been faid concerning the fun, is alike applicable to the moon, which we may put in the place of the fun; the moon's orbit in the place of the ecliptic; and the time of the moon's revolution round the earth in the place of the revolution of the fun round the earth: And we fhall find the motion of the earth's pole parallel to the lunar orbit, which is always the fame at every quarter of the time of the revolution of the moon round the earth, and the ofcillation of the earth's axe to the plane of the lunar orbit, which is completed, in each femirevolution of the moon round the earth.

But whereas the plane of the lunar orbit, which is always inclined to the plane of the ecliptic in an angle of about 5 degrees, never continues in a conftant pofition, like the plane of the ecliptic, fo that its pole defcribes a fmall circle parallel to the ecliptic, at the diftance of about 5 degrees from the pole thereof; it follows, that the preceffion, with refpect to the lunar orbit, is not the fame as with refpect to the ecliptic; and that the motion of the pole parallel to the lunar orbit fhould be referred to the plane of the ecliptic: Which is done by refolving the motion of the pole, parallel to the plane of the lunar orbit, into two motions, the one parallel to the plane of the ecliptic, and the other perpendicular thereto, and in the plane of the folfticial colure.

The

The former of theſe two motions gives the preceſſion with reſpect to the ecliptic, and has its direction always the ſame way.

The latter motion has two oppoſite directions, in the two ſemirevolutions of the pole of the lunar orbit round the pole of the ecliptic, and cauſes an oſcillation of the terreſtrial axe on the plane of the ecliptic, which is compleated in a revolution of the pole of the lunar orbit round the pole of the ecliptic.

From all that has been ſaid, it follows, that there are five diſtinct motions of the pole of the earth; namely, two of preceſſion, which are parallel to the plane of the ecliptic, and three of oſcillation on the plane of the ecliptic.

The two of preceſſion are cauſed, the one by the ſun, the other by the moon. That, which is cauſed by the ſun, is conſtantly the ſame at every quarter of the time of the revolution of the ſun round the earth, that is, every three months: That which is cauſed by the moon, is conſtantly the ſame at every quarter of the time of the revolution of the moon round the earth; that is, about every ſeven days.

Of the three motions of oſcillation, one is cauſed by the ſun, and is completed in the time of the ſemirevolution of the ſun round the earth, taken from one equinox to the following one; that is, in ſix months.

The other is cauſed by the moon; and each oſcillation is completed in the ſpace of a ſemirevolution of the moon round the earth; that is, in about 14 days.

The third is cauſed likewiſe by the moon, and ariſes from the plane of her orbit being different from the plane of the ecliptic, and from the pole of the lunar orbit making its revolution about the pole of the ecliptic

ecliptic in about 18 years and two thirds. And this oſcillation is compleated in the time of the revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about 18 years and two thirds.

It will appear in the memoir, that there is a relation purely geometrical between the quantity of the nutation, during the time of the ſemirevolution of the pole of the lunar orbit, and the quantity of the preceſſion, cauſed likewiſe by the moon in the ſame time. This relation is quite independent of the force of the moon, of the quantity of the earth's flatneſs, of the quantity of the terreſtrial matter, and, in a word, of every thing of a phyſical nature that can enter into the problem.

We are content to examine the motions of the pole of the earth produced by the ſun and the moon. The ſame method, and the ſame *formulæ*, will give likewiſe the motions of the terreſtrial pole ariſing from any other planet, as Saturn, Jupiter, &c. but theſe motions are too minute to merit attention.

Whatever has been ſaid of the action of the ſun on the redundant matter about the earth's equator, is alſo applicable to his action on a ſimple ring placed at the equator, without adhering to the terreſtrial globe; and the motion of the pole of ſuch ring may be determined by the ſame method, and conſequently the motion of its nodes on the plane of the ecliptic, and the alteration of the inclination of its axe to the ſame plane. And ſince theſe motions are the ſame, whether the ring be ſuppoſed entire, or a ſmall portion of it only be conſidered, or a mere point thereof, the motions of the nodes, and the alteration of the inclination of a moon, or a ſatellite of a planet, may

thereby

thereby be known. And the *formulæ* differ in nothing from those of the motion of the nodes of the earth's equator, and of the alteration of the obliquity of the earth's axe to the plane of the ecliptic, but in this; that the action of the sun on the ring to make it turn, is exerted entirely thereon; whereas in the problem of the precession this force must necessarily be distributed throughout the whole mass of the earth, on account of the adherence of the ring to the globe of the earth.

DIVISION of the WORK.

This memoir is divided into four sections.

The 1st section treats of the motion of the pole of the terrestrial equator caused by the sun.

The 2d section treats of the motion of the pole of the terrestrial equator caused by the moon.

The 3d section treats of the motion of the pole of a ring, or of the orbit of a moon, caused by the sun.

The 4th section contains the application of the *formulæ* found in the other sections.

SECT. I.

Of the motion of the pole of the terrestrial equator caused by the action of the sun.

PROBLEM I.

Article 1. *To find the moment, which results from the attraction of each particle of the earth towards the sun, in the inverse ratio of the square of the distance, to make the earth's axis turn upon its centre* C.

Let *E F E′ F E*, *fig.* 1. *n°* 1. be the section of the earth by the plane of the circle of the sun's declination;

tion; $PEP'E'P$ the circumference whose diameter is the earth's equator, EE' or PP'. Let EE' be perpendicular to PP', and in the equator. Let II' be perpendicular to CS, and assume this diameter II' for the leaver to which all the moments are to be referred.

The motion of any point, as g, towards the sun, is $\frac{S}{\overline{Sg}^2}$. By resolving this motion into two, one according to gC, the other parallel to CS; the motion according to gC has no tendency to make the point g turn round the centre C. The motion of the point g parallel to CS will be $\frac{S}{\overline{Sg}^2} \times \frac{SC}{Sg}$ or $\frac{S \times SC}{\overline{Sg}^3}$; but the motion of the centre C according to CS is $\frac{S}{\overline{SC}^2}$; therefore the relative motion of the point g, in regard gard to the centre C, is $\frac{S \times SC}{\overline{Sg}^3} - \frac{S}{SC}$ or $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right)$.

The moment of the point g to turn about the centre C, will therefore be $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right) \times g \times CI$, the moment being the product of the motion by the mass of the body, and by the arm of the leaver. And this moment causes the point I to approach towards the sun, when it is positive, or when SC is greater than Sg; and causes the point I to recede from the sun when it is negative, or when Sg is greater than SC.

It will be found, in like manner, that the moment of the point g' to turn about the centre C, is $S \times SC \times \left(\frac{1}{\overline{Sg'}^3} - \frac{1}{\overline{SC}^3}\right) \times g' \times CI'$; or with regard to the point I which is on the opposite side of the centre to the point I', this moment, by changing the signs, will be, $S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g' \times CI' = S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g \times CI$. Therefore the moment of the two equal points g and g' to cause the point I to turn about the centre C, is $S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right) \times g \times CI + S \times SC \times \left(\frac{1}{\overline{SC}^3} - \frac{1}{\overline{Sg'}^3}\right) \times g \times CI = g \times CI \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$.

If the points γ and γ' be taken equally distant from the points D and D', as the points g and g', it will likewise be found, that the moment of the equal points γ and γ' to cause the point I to turn about the centre C, is $\gamma \times \overline{CI} \times S \times \overline{SC} \times \left(\frac{1}{\overline{S\gamma'}^3} - \frac{1}{\overline{S\gamma}^3}\right) = \gamma \times \overline{CI} \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$.

Therefore the moment of all the four points together, g, g', γ, γ', to cause the point I to turn about the centre C, is $(g - \gamma) \times \overline{CI} \times S \times SC \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{Sg'}^3}\right)$, which

which becomes $=$ o when the point g can be taken $=$ γ, as may be done throughout the whole extent of the circle $P E P' E' P$; but if the zone $E P' E' P E F' E' F E$ be taken away from this circle, to have the true figure of the earth, we shall have $g = \overline{gf} \times \overline{g r}$, and $\gamma = - \overline{\gamma f} \times \overline{\gamma \varrho}$, taking the points g and γ for the elements of the zone, and $g - \gamma = \overline{gf} \times \overline{\overline{\gamma \varrho} - \overline{g r}}$, because $\overline{\gamma f} = \overline{gf}$.

Let the lines $C S$ and $C P$, or $C E$, be called s and a, respectively; the radius 1, or unity; the sine of the angle $S C P$, V; its cosine, or the sine of the angle $S C E$, u; $C K$, x; $g K$, y; $\overline{g m}$, z; $\overline{\gamma M}$, z'; $\overline{gf}$ or $\overline{\gamma f'}$, $d u$; $P F$, a a; we shall have $g n =$ a z; $\gamma v =$ a z' (by the property of the ellipse); also calling $\overline{g r}$, r; and $\gamma \varrho$, ϱ; and regarding the little triangle $g r n$ as a right-lined one, and similar to the triangle $g C m$, we shall have $r = \frac{z}{a} \times$ a z, and likewise $\varrho = \frac{z'}{a} \times$ a z'.

By the proportion of sines to the sides of triangles, we shall have γ G $= y V$: $C N = x u$. Finally, if we consider the point S, or the sun, as at an infinite distance, we shall have $S g = S K = S C - C K = s - x$, and $S g' = S K' = S C + C K' = s + x$. Therefore $\frac{1}{Sg^3} = \frac{1}{s^3 - 3s^2x + 3sxx - x^3}$, and $\frac{1}{Sg'^3} = \frac{1}{s^3 + 3s^2x + 3sx^2 + x^3}$. Therefore $\frac{1}{Sg^3} - \frac{1}{Sg'^3} =$

$\frac{6s^2x+2x^3}{(s-x)^3\times(s+x)^3}$; and, rejecting the infinitely ſmall magnitudes, $=\frac{6x}{s^4}$.

Therefore the moment wherewith the four points g, g', γ, γ', cauſe the point I to turn, will, by ſubſtituting in the place of the letters their analytical values, be $(\varrho\, du - r\, du)\times S\times y\times\frac{6x}{s^3}$; or putting, inſtead of ϱ and r, the values above found, we ſhall have for the moment $\left(\frac{\mathrm{a}}{a}z'z'-\frac{\mathrm{a}}{a}zz\right)\times du\times S\times y\times\frac{6x}{s^3}=\frac{6\,\mathrm{a}\,S}{s^3a}\times y\,x\,d\,u\times(z'+z)\times(z'-z)$. But $z'+z=\gamma M+gm=2\,\overline{\gamma G}=2\,y\,V$; and $z'-z=\gamma M-g\,m=2\,\overline{GM}=2\,\overline{CN}=2\,x\,u$. Therefore $(z'+z)\times(z'-z)=4\,V\,u\,x\,y$; and the moment is $\frac{6\mathrm{a}S}{s^3}\times4\,V\,u\times x\,x\,y^2\,du=\frac{6\mathrm{a}S}{s^3}\times4Vu\times x\,x\,d\,x\,u$. But in the circle $D\,g\,I$ we have $y\,d\,u=a\,d\,x$, and the moment is $\frac{6\mathrm{a}S}{s^3}\times4Vu\times x\,x\,y\,d\,x=\frac{6\mathrm{a}S}{s^3}\times4\,V\,u\times dx\sqrt{aa-xx}$. But the integral of $x^2\,d\,x\sqrt{aaxx}$ is $\frac{1}{4}aa\int dx\sqrt{aa-xx}-\frac{x}{4}(aa-xx)^{\frac{3}{2}}$, and when $x=a$, this integral is $\frac{1}{4}aa\int dx\sqrt{aa-xx}$, or, calling the ratio of the circumference to the diameter π, we ſhall have $\int d\,x\sqrt{aa-xx}$, or the area of the quadrant whereof a is the radius, $=\frac{1}{2}a\pi\times\frac{1}{2}a=\frac{1}{4}a^2\pi$. Therefore the integral of $\frac{6a'S}{s^3}\times4\,V\,u\times x\,x\,d\,x$

$\sqrt{aa}$

$\sqrt{aa-xx}$, or the ſum of the moments of all the points which compoſe the elliptic zone $E\,P'\,E'\,P\,E\,F\,E\,F\,E$ is $\frac{6aS}{s^3} \times \frac{Vu\pi a^4}{4}$.

If the terreſtrial ſpheroid be cut by any plane parallel to the plane $P\,E\,P'\,E'\,P$ of the circle of declination, *fig.* 1. *n°* 2. the ſection will be an ellipſe ſimilar to the ellipſe $E\,F'\,E'\,F\,E$; and if the greater ſemiaxe $c'\,L$ of this ellipſe be called X, it will be found, as has been already in the ellipſe $E\,F'\,F'\,F\,E$, that the ſum of the moments of all the points of this ellipſe to turn about its centre C, is $\frac{6aS}{S^3} \times \frac{Vu\pi X^4}{4}$.

It has not been taken into conſideration, that here the centre S of the ſun is a little below the plane of the ſection, the line $c'\,S$ making an inſenſible angle with the line $C\,S$.

Calling $C\,c'$, Y; the moment of all the ſections parallel to $E\,F'\,E'\,F\,E$ to turn about the axe $C\,c'$ will be the integral of $\frac{3aS}{S^3} \times \frac{Vu\pi X^4}{2} \times d\,Y$, or the integral of $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times (aa\,YY)^2 \times d\,Y = \frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times$ $(a^4\,d\,Y - 2a^2\,Y^2\,d\,Y + Y^4\,d\,Y)$, and this integral is $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times (a^4\,Y - \frac{2}{3}\,a^2\,Y^3 + \frac{1}{5}\,Y^5)$; and when $Y = a$, the integral is $\frac{3aS}{s^3} \times \frac{Vu\pi}{2} \times \frac{8}{15}\,a^5$. This is the ſum of the moments of all the points of the hemiſpheroid of the earth formed by the ſection $E\,P'\,E'\,P\,E$ of the circle of declination, and the ſum of the moments of all the points of the whole ſpheroid

roid to turn about the axe $\mathcal{Q}\mathcal{Q}'$ is $\frac{3aS}{s^3} \times Vu\pi \times \frac{8}{15}a^5$. Which was to be found.

COROLLARY.

2. If it were required to find the ſum of the moments of all the points of a ſingle crown $ELQE'Q'E$, *fig.* 1. n^o 2. placed at the equator, and detached from the earth, to turn about the axe $\mathcal{Q}\mathcal{Q}'$ of the circle of the ſun's declination, the thickneſs of the crown being a a:

The moment of the point L to turn about c' is the ſame as that of the point γ to turn about C; as is manifeſt from the reſolution of the motion according to SL into two, $L\gamma$, and γS*; and from the great diſtance of the ſun γS will not differ from LS; the triangle $L\gamma S$ being right-angled at γ.

By the preceding problem, if $c'L$, or CY, be called X; and Cc', Y; and all the other denominations of that problem be retained; we ſhall have $\gamma\varkappa = Xu$, and $C\varkappa = XV$.

And the moment of the point γ is $\gamma \times \gamma\varkappa \times S \times \overline{SC} \times \left(\frac{1}{\overline{S\gamma}^3} - \frac{1}{\overline{SC}^3}\right) = \gamma \times Xu \times S \times \frac{3XV}{s^3}$.

And putting inſtead of γ, the element of the crown, which is a $a \times du$, calling, in this caſe, du the element of the circumference $ELQE'$, the moment of an element of the crown will be a$a \times \frac{3S}{s^3} \times Vu \times$

* The lines SL and γS are not drawn in the *fig.* to avoid confuſion.

$XXdu$

$XXdu$. But in the circle $EQE'E$ we have, $Xdu = a\,dY$; therefore the moment of an element of the crown is a$a \times \frac{3S}{s^3} \times Vu \times a \times dY = \frac{3S}{s^3} \times$ a$a^2\,Vu \times dY\sqrt{aa - YY}$; and the sum of the moments of the quadrant ELQ of the crown is $\frac{3S}{s^3} \times$ a$a^2\,V\,u \times \frac{1}{2}a\pi \times \frac{1}{2}a = \frac{3S}{s^3} \times$ a$Vu \times \frac{1}{4}\pi a^4$. And every other quadrant of the crown having an equal motion, the sum of the moments of all the points of the crown, to turn, in vertue of the solar action, about the axe QQ', is $\frac{3S}{s^3} \times$ a$Vu \times \pi a^4$.

PROBLEM II.

3. *To find the sum of the moments of all the points of the terrestrial spheroid turning about the axe* QQ′, *the motion of the point* E *being given, and* $= \mu$.

We must first seek the sum of the moments of all the points of the ellipse $EF'E'FE$, *fig.* 1. *n*° 2. turning about its centre C; and to have the sum of the moments of all the points of this ellipse, we shall first find the sum of the moments of all the points of the circle $EP'E'PE$, and then we shall subduct therefrom the sum of the moments of the whole elliptic zone $EP'E'PEF'E'FE$.

The motion of any point of the circle $EP'E'PEC$, placed at the distance x from the centre C, will be $\frac{x}{a}\mu$; the motions being here in the same ratio as the di-

ſtances from the centre, ſince all the points complete their revolution in the ſame time; and the moment of all the points which compoſe the circumference $2\pi x$ will be $2\pi x \times \frac{x}{a} \mu \times x$; the moment being produced from the maſs by the motion, and by the arm of the leaver.

Therefore the motion of all the concentric circumferences which compoſe the circle, is $\int \frac{2\pi\mu}{a} \times x^3 dx$ $= \frac{1}{2}\frac{\pi\mu}{a} \times x^4 = \frac{1}{2}\frac{\pi\mu}{a} \times a^4$, when $x = a$, as for the circle $EP'E'PE$.

To have the ſum of the moments of all the points of the elliptical zone $EP'E'PEF'E'FE$, all the points of the zone may be conſidered as placed at the diſtance a, from the centre C, and having, conſequently, the ſame motion as the point E, becauſe the greateſt thickneſs PF of the zone is very ſmall.

The ſum of the moments of the zone will be equal to the quantity of matter multiplied by the motion μ, and by the arm of the leaver a, or $2a\pi \times \frac{1}{2} \text{a} u \times \mu \times a = \pi\mu \text{a} a^3$.

Therefore the ſum of the moments of all the points of the ellipſe $EF'E'FE$ turning about its centre C, the motion of the point E being μ, will be $\frac{1}{2}\frac{\pi\mu}{a} \times a^4 - \pi\mu \text{a} a^3$.

It will likewiſe be found generally, that the ſum of the moments of a ſimilar ellipſe, of which $C'L$, or X, is half the greater axe, and which turns about its centre C', is $\frac{1}{2}\frac{\pi\mu}{a} \times X^4 - \frac{\pi\mu \text{a}}{a} \times X^4$.

Therefore

Therefore the ſum of all the ſimilar ellipſes which compoſe the terreſtrial hemiſpheroid, is $\int \frac{1 - 2a}{2a} \times \pi \mu \times X^4 d Y = \frac{1 - 2a}{2a} \times \pi \mu \times \int (aa - YY)^2 \times dY$, which, as has been ſeen at the end of *Prob.* I. when, after the integration Y has been made $= a$, will be $\frac{1 - 2a}{2a} \pi \mu \times \frac{8}{15} a^5$, and multiplying by 2, the ſum of all the points of the terreſtrial ſpheroid will be $(1 - 2a) \times \pi \mu \times \frac{8}{15} a^4$. Which was to be found.

COROLLARY.

4. If you would find the ſum of the moments of all the points of the ring $E Q E' Q' B$, *fig.* 1. *n*° 2. turning about the axe $Q Q'$:

The moment of an element L of the crown placed at the diſtance X from the axe of rotation $Q Q'$, is $a\, a\, d u \times \frac{\mu X}{a} \times X$, which is the product of the quantity of matter by the motion, and by the arm of the leaver. But in the circle $E Q E' E$ we have, $d u \frac{a d Y}{X}$; therefore the moment of an element of the crown is $a\, a \mu \times X d Y$, whoſe integral, which is $a\, a \mu \times a\, a \pi$, gives the ſum of the moments of all the points of the crown. Which was to be found.

Problem III.

5. *Having given, the moment wherewith the earth turns about the axe* QQ′ *of the circle of the sun's declination, To find the motion of the pole* P, *or, which is the same, of the point* E *in the plane of the circle of the sun's declination.*

By *Prob.* I. the moment wherewith the earth turns about the axe $C\,Q$, by the action of the sun, is $\frac{3\,S}{s^3} \times \mathrm{a}\,\pi\,\nu\,u \times \frac{8}{15}\,a^5$.

By *Prob.* II. the moment wherewith the earth turns about the same axe $C\,Q$, the velocity of the point P, or of the point E, being μ, is $(1-2\mathrm{a}) \times \pi\,\mu \times \frac{8}{15}\,a^4$.

We shall have then $\frac{3\,S}{s^3}\,\mathrm{a}\,\pi\,\nu\,u \times \frac{8}{15}\,a^5 = (1-2\mathrm{a}) \times \pi\,\mu \times \frac{8}{15}\,a^4$; whence we get $\mu = \frac{3\,S\,\mathrm{a}\,a\,\nu u}{s^3 \times (1-2\mathrm{a})}$. And this is the motion of the pole P, or of the point E, about the centre C. Which was to be found.

Corollary.

6. In like manner, the moment wherewith the crown $E\,Q\,E'\,Q'\,E$ turns about the axe $Q\,Q'$ being given, the motion of the point E may be known. For,

By the *Coroll.* of *Prob.* I. the moment of this crown, turning about the axe $C\,Q$ by the action of the sun, is $\frac{3\,S}{s^3} \times \mathrm{a}\,\pi\,\nu\,u \times a^4$.

By

By the *Coroll.* of *Prob.* II. the moment of this crown is alſo a $a\mu \times a^2 \pi$.

We have then $\frac{3S}{s^3} \times 2\pi\nu u \times a^4 = 2a\mu \times a^2\pi$; whence we get $\mu = \frac{3S}{s^3} \times a\nu u$. Which was to be found.

Remark. It is proper to obſerve, that the motion of the point *E* of the crown is the ſame, whether the crown be entire, or there be no more of it but the point *E*.

For, by *Prob.* I. the motion of any point *g*, parallel to *C S*, *fig.* 1. *n°* 2. and reſpectively to the centre *C*, is $S \times \overline{SC} \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3}\right)$, which is $= S \times \frac{3x}{S^3}$; and when the point *g* becomes the point *E*, we have $x = a\nu$, ν being the coſine of the angle *D C E*. Therefore the reſpective motion of the point *E* parallel to the motion *C S* of the centre *C*, is $S \times \frac{3a\nu}{S^3}$; and reſolving this motion into two, the one according to *E C*, and the other perpendicular to *E C*, this latter will be $= S \times \frac{3a\nu}{s^3} \times \frac{au}{a} = \frac{3S}{s^3} a\nu u$, being the ſame as that which was found for the point *E* when it is united to the crown.

This ſhews that the motion of the other points of the crown produce no alteration in the motion of the point *E*, which is owing to this; that all the points of the crown have the ſame angular motion about the axe *C Q*.

PROBLEM IV.

7. *To find at every inſtant the variation of the place of the pole of the earth.*

The earth has two motions of rotation, the one about the axe of the equator, which is likewiſe the earth's axis; and the other about the axe $C\,Q$ of the circle of the ſun's declination; this motion is cauſed by the ſun's action on the redundant matter about the equator.

The point E, *fig.* 1. *n°* 2. which is the interſection of the circumference $E\,Q\,E'\,Q'\,E$ of the equator and the circle $P\,E\,P'\,E'\,P$ of the ſun's declination, will therefore have at every inſtant two motions, whoſe directions are perpendicular to each other.

The former of theſe two motions, which is the diurnal motion, is uniform; and if we call it m, the ſpace run through in an inſtant, $d\,t$, is $m\,d\,t$; the motion being always equal to the ſpace divided by the time, or the ſpace equal to the product of the time. Alſo let $E\,e$ be that ſpace.

The ſecond motion of the point E which is performed in the circumference $E\,P\,E'\,P'\,E$ of the circle of the ſun's declination, and ariſes from the action of the ſun, is continually accelerated, from the continual application of the ſun's action; and if we call the initial motion μ, at the firſt term of the inſtant, $d\,t$, or the increment of the motion, the motion at the end of the inſtant $d\,t$, is $\mu\,d\,t$; and the ſpace gone thro' uniformly by the motion during the inſtant $d\,t$, is $\mu\,d\,t \times d\,t$, or $\mu\,d\,t^2$. Let $E\,\varepsilon$ be that ſpace.

The

The point E, in consequence of these two motions together, $E e$ and $E \varepsilon$, circulates neither in the circumference $E Q E' Q' E$ of the equator, nor in the circumference $E P E' P' E$ of the sun's declination. But if we form the rectangle $E e e' \varepsilon E$, the diagonal $E e'$ will be the elementary arc of the circumference $E e' q E' E$ wherein the point E will circulate; and the angle $e E e'$, equal to the angle $Q c q$, equal to the angle $P' C p'$, will be the angle whereby the pole P' is elevated, and the pole P depressed, below the circle of declination, by moving in the circumference $P' p' Q q P p P'$, whose plane is perpendicular to the plane of delination; and the point p' is the true place of the pole at the end of the instant $d t$, and the small arc $P' p'$ expresses the instantaneous variation of the place of the earth's pole. Which was to be found.

COROLLARY I.

8. The similar sectors $e E e'$, $Q C q$, or $P' C p'$, whose three sides of the one are each parallel to the three sides of the other, give this proportion, $E e$: $e e'$ or $E \varepsilon$ :: $C P'$: $P' p' = \frac{E \varepsilon}{E e} \times C P' =$ (by the preceding *Prob.*) to $\frac{\mu \, d t^2}{m \, d t} \times C P' = \frac{\mu \, d t}{m} \times a$. But by *Problem* III. $\mu = \frac{3 S \text{a} a \nu u}{s^3 (1 - 2\text{a})}$; therefore $P' p' =$

$$\frac{3 S \text{a} a \nu u}{s^3 \times m \times (1 - 2\text{a})} \times a \, d t = A \nu u a \, d t, \text{ making } A$$

$$= \frac{3 S \text{a} a}{s^3 \times m \times (1 - 2\text{a})}.$$

COROL-

Corollary II.

9. If, inſtead of the terreſtrial ſpheroid, we ſuppoſed only a ſingle ring, it would be found in like manner, that the pole P of ſuch ring would ſhift its place during the inſtant $d\,t$, by running through the arc $P'\,p' = \frac{E\,\varepsilon}{E\,e} \times CP' = \frac{\mu\,d\,t}{m} \times a$. But then we ſhould take the value of μ as in *Coroll.* of *Prob.* III. which is $\frac{3\,S}{s^3} \times a\,\nu\,u$. Therefore we ſhall have $P'\,p' = \frac{\mu\,d\,t}{m} \times a = \frac{3\,S\,a}{s^3 \times m} \times \nu\,u\,a\,d\,t$.

To apply, then, the *formula* of the preced. *Coroll.* it will ſuffice to make $\frac{a}{1 - 2a}$ in that *Coroll.* $= 1$. for then the *formula* of *Coroll.* I. becomes the ſame with that juſt now found in this *Coroll.*

This will be of uſe hereafter for applying the *formulæ* of the motion of the nodes of the terreſtrial equator to the motion of the nodes of a moon or ſatellite of a planet.

Corollary III.

10. It ſhould be well obſerved in the preceding *Coroll.* that the direction $P'\,p'$ of the pole P' is ever parallel to the direction $E\,e$ of the point E, of the equator, ſince they are both perpendicular to the plane of the circle of the ſun's declination, and the direction $P'\,p'$ goes on the ſame ſide as the direction $E\,e$ of the point E, which is placed on the ſame ſide of the

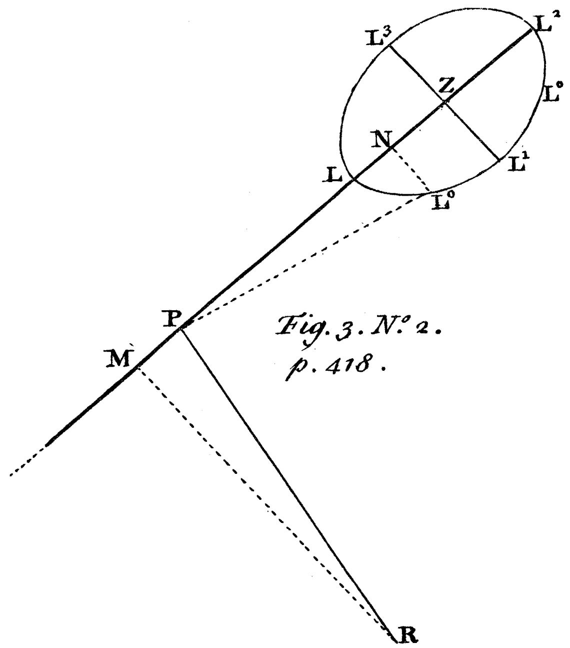


Fig. 3. N.º 2.
p. 418.

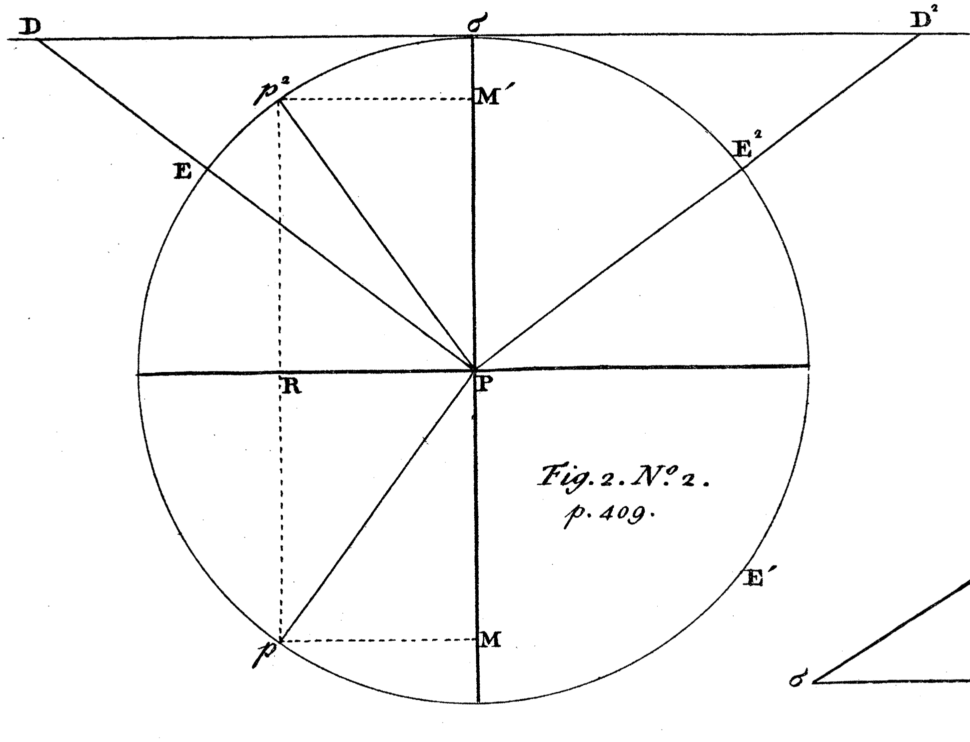
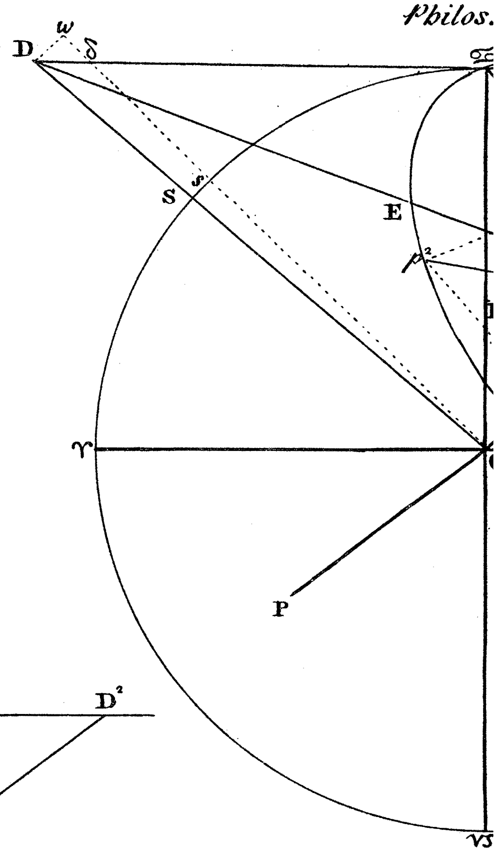


Fig. 2. N.º 2.
p. 409.



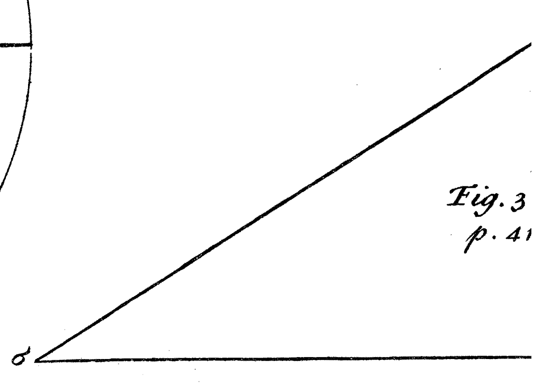


Fig. 3
p. 41

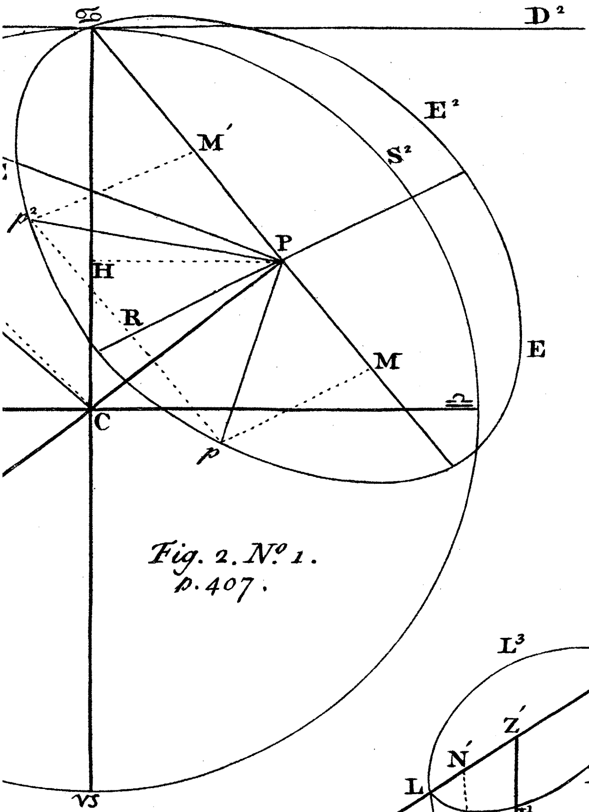


Fig. 2. N.° 1.
p. 407.

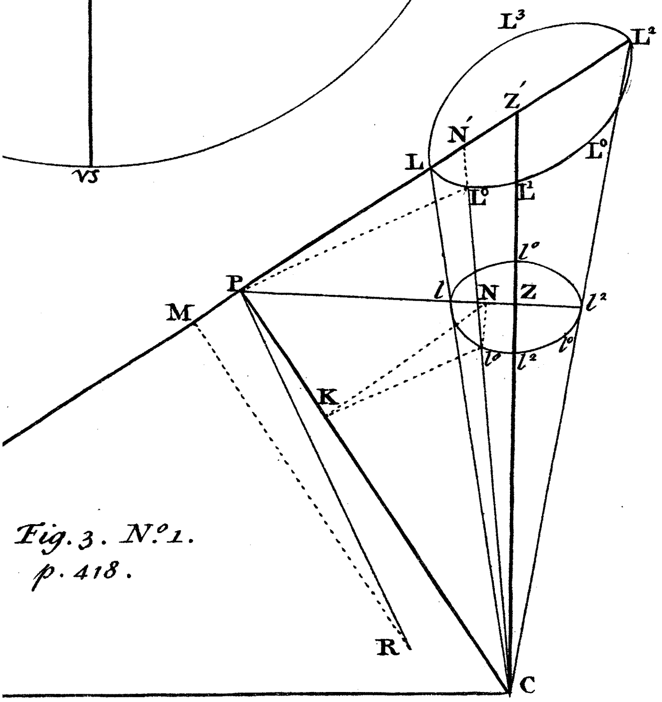


Fig. 3. N.° 1.
p. 418.

J. Mynde sculp.

the pole P', with refpect to the line $S\ C\ D'$, and the direction of the oppofite pole P, goes on the fame fide as the direction of the point E of the equator which is placed on the fame fide of the pole P with refpect to the line $S\ C\ D'$. This fhould be carefully attended to in regard to the following *Problem*.

PROBLEM V.

11. *To determine at every inftant the motion of the pole of the earth with refpect to the ecliptic.*

$P\ P'$, *fig.* 2. *n°* 1. *&* *n°* 2*. is the earth's axe; ♈ ♋ ♎ ♑ reprefents the ecliptic; ♋ $E\ p\ E$ ♋ is the tangent plane to the pole P, and parallel to the equator; the line D ♋ D^2 is the interfection of this plane with that of the ecliptic, and is tangential at the common point ♋ to the circles ♋ $E'\ p\ E$ ♋ and ♈ ♋ ♎ ♑. The angle $P\ C$ ♋ is the angle of inclination of the earth's axe to the plane of the ecliptic, being likewife the folfticial angle. The earth, by its diurnal motion, turns about the axe $P'\ P$ according to the order of ♋ $E'\ p\ E$ ♋; and the fun, by the annual motion in the ecliptic, moves according to the order of ♈ ♋ ♎ ♑.

Let S be any place of the fun in the ecliptic. By *Art.* 10. the motion of the pole P making the acute angle $P\ C\ S$, will be according to $P\ p$ perpendicular to the plane $P\ C\ S\ D\ P$, and confequently to the line $P\ D$.

---

* *Note*, That the *fig.* *n°* 2. is only to reprefent the circle ♋ $E\ p$ ♋, which could not be reprefented but by projection in *fig.* *n°* 1.

5 Forming,

Forming, therefore, the rectangle $P\,M\,p\,R$, the motion of the pole according to $P\,p$ will be resolved into two, according to $P\,R$ and $P\,M$.

The motion of the pole according to $P\,R$ is parallel to the plane of the ecliptic, and to the line of the nodes ♈ C ♎. This motion tends to move the pole P about the axe of the ecliptic, and in a parallel to the ecliptic, and causes the equinoctial point ♈ to recede, and the point ♑, and the node ♎, to advance towards the sun; which produces a precession of the equinox equal to the angular motion of the pole P about the axe of the ecliptic; for it is indifferent whether the pole P of the earth moves about the axe of the ecliptic, whilst the plane ♈ ♋ ♎ ♑ remains fix'd; or the same be supposed to be fix'd, and the plane of the ecliptic ♈ ♋ ♎ ♑ to turn about its centre C, and the axe of the ecliptic.

If we take another place S^2 of the sun at the same distance from the solstice ♋, but on the side contrary to S, the direction of the pole P will be, by *Art.* 10. according to Pp^2, perpendicular to the plane PCS^2D^2P, and consequently to the line $P\,D^2$; and the sun's declination being the same in D^2 as in D, the motion $P\,p^2$ will be equal to the motion $P\,p$.

Resolving the motion according to $P\,p^2$ into two, according to $P\,K$ and $P\,M'$, the motion according to $P\,R$ produces the same precession, and with the same sign, or the same direction, as when the sun was on the other side of the solstice ♋, and in D; and the precession is ever the same at every quarter revolution of the sun.

The motion according to $P\,M'$ causes the same variation in the inclination of the earth's axe to the ecliptic,

ecliptic, as when the ſun was on the other ſide in S, but with a contrary ſign, or direction; and the ſum of the lines PM', from the ſolſtice ♋ to ♎, will be the ſame as the ſum of the lines PM, from ♈ to ♋, but with a contrary ſign: ſo that the angle of inclination of the earth's axe to the plane of the ecliptic is continually increaſing from ♈ to ♋, and then decreaſing from ♋ to ♎; and as the diminution is equal to the increaſe, the inclination becomes again the ſame in ♎ as it was in ♈, having undergone an oſcillation whoſe period is ſix months.

Whilſt the ſun paſſes from ♎ to ♑, there are the ſame motions of the pole, as whilſt it was paſſing from ♋ to ♎.

Thus is the motion of the pole determined at every inſtant, with reſpect to the ecliptic. Which was to be done.

Problem VI.

12. *To find the quantity of the preceſſion, or, which is the ſame, the arc of the parallel to the ecliptic run through by the pole of the earth* P *in the ſpace of time the ſun is paſſing from the equinox to the ſolſtice, or from the ſolſtice to the equinox; which may be taken for a quarter of the time of the ſun's revolution about the earth; the angle comprehended between the equinox and the enſuing ſolſtice differing but by an infinitely ſmall quantity from 90 degrees.*

Calling, as in *Prob.* I. the ſemi-axe CP, a; *fig.* 2. *n°* 1. & *n°* 2. the ſine of the angle PCS, or PCD, V; and its coſine, or the ſine of the ſun's declination, u:

Let $P\text{☊}$, be b; $C\text{☊}$, r; we shall have $CD = \frac{a}{u}$; $PD = \frac{a}{u}V$; the difference Ss of the arc ♈ S, which is the sun's distance from the node ♈, will be $\frac{rr\times d\overline{CD}}{\overline{CD}\times\overline{\text{☊}D}}$, as is found by these two proportions, $r : \overline{Ss} :: CD : D\omega = \frac{CD\times\overline{Ss}}{r}$, and $D\omega = \frac{CD\times\overline{Ss}}{r} : \delta\omega = d\overline{CD} : \overline{C\text{☊}} = r : \overline{\text{☊}D}$; whence we have $\overline{CD}\times\overline{Ss}\times\overline{\text{☊}D} = rr\times d\overline{CD}$, and $\overline{Ss} = \frac{rr\times d\overline{CD}}{\overline{CD}\times\overline{\text{☊}D}} = \frac{r\,du}{\sqrt{\frac{aa}{uu}-rr}}$ (putting instead of $\overline{CD}$ and $\overline{SD}$, their values $\frac{a}{u}$ and $\sqrt{\frac{aa}{uu}-rr}$) $=$ to the space run through by the sun S in the ecliptic in an instant, $= 2\pi r\times\frac{dt}{T}$, calling T the time the sun employs in running thro' the whole circumference $2\pi r$.

Whence we have $dt = \frac{T}{2\pi r}\times\frac{r\,du}{\sqrt{\frac{aa}{rr}-uu}} = \frac{T}{2\pi}\times\frac{du}{\sqrt{\frac{aa}{rr}-uu}}$.

By

By *Art.* 8. we have $a\,P\,p = A\,V\,u \times a\,d\,t$; therefore, by *Prob.* V. the instantaneous preceffion will be $A\,V\,u \times a\,d\,t \times \frac{\overline{PR}}{Pp}$, or $A\,V\,u \times a\,d\,t \times \frac{\overline{P\,♋}}{\overline{PD}}$, the triangles $P\,p\,r$, $P\,D\,♋$, being fimilar, fince the three fides of the one are perpendicular to the three fides of the other. Putting, therefore, inftead of $\overline{PD}$, $\overline{P♋}$, and $d\,t$, their values above found, we fhall have the inftantaneous preceffion $= A\,V\,u \times \frac{a\,b}{\frac{aV}{u}} \times \frac{T}{2\pi} \sqrt{\frac{aa}{rr} - uu}$

$= A\,b \times \frac{T}{2\pi} \times \frac{u\,u\,d\,u}{\sqrt{\frac{aa}{rr} - uu}}$. But $\int \frac{u\,u\,d\,u}{\sqrt{\frac{aa}{rr} - uu}} = -$

$u\sqrt{\frac{aa}{rr} - uu} + \int du \sqrt{\frac{aa}{rr} - uu}$, or fimply $= \int \frac{u\,u\,d\,u}{\sqrt{\frac{aa}{rr} - uu}}$

when $u = \frac{a}{r}$, as it happens when the fun is in the folftice ♋, and then $\int d\,u \sqrt{\frac{aa}{rr} - uu}$, which is the area

of a quadrant of a circle whofe radius is $\frac{a}{r}$, will be

$\frac{a\pi}{2\,r} \times \frac{a}{2\,r} = \frac{aa\pi}{4\,r\,r}$.

Therefore the preceffion, whilft the fun is paffing from ♈ to ♋, is $A\,b \times \frac{T}{2\pi} \times \frac{aa\pi}{4rr} = A\,b \times \frac{T}{2} \times \frac{a\,a}{4\,r\,r}$;

 and

and calling CH, which is = the radius of the circle parallel to the ecliptic which the pole P runs thro', g; we ſhall have $g = \frac{aa}{r}$, and the preceſſion $Ab \times \frac{T}{2} \times \frac{aa}{4rr}$ will be likewiſe $Ab \times \frac{T}{2} \times \frac{g}{4r}$. Which was to be found.

And this alſo is the quantity of the preceſſion during the time the ſun paſſes from the ſolſtice to the equinox, or generally, during the time of each quadrant of the ſun's revolution; as was ſeen in *Prob.* V.

Remark.

13. The difference of the ſun's diſtance S ♈ from the equinoctial point, is equal to the arc the ſun goes through in the ecliptic, when the node ♈ is fix'd, as was ſuppos'd in the foregoing *Prob.* but when the node ♈ moves in the ecliptic the contrary way to the ſun's motion, the difference of the ſun's diſtance from the node is equal to the arc gone through by the ſun, increaſed by the arc gone through by the node ♈.

Therefore always taking Ss, *fig.* 2. *n°* 1. for the difference of the ſun's diſtance from the node, we ſhall have $\overline{Ss}$ equal to the arc deſcribed by the ſun, more by the arc deſcribed by the node.

But, by the former *Prob.* we have ſeen, that the arc run through by the pole P in an inſtant dt, and in a parallel to the ecliptic, whereof g is the radius, is

$$AVu \times adt \times \frac{\overline{P ♋}}{PD} = AVu \times \frac{ab}{\frac{aV}{u}} \times dt = Abuudt.$$

The arc run through by the node ♈ will be then

Abu

$A\,b\,u\,u\,d\,t \times \frac{r}{S}$, since these arcs, which measure equal angles, are as their radii g and r.

We shall have, therefore, in the above *Problem*, $\overline{Ss}$, or $\frac{r\,du}{\sqrt{\frac{aa-uu}{rr}}}, = 2\,\pi\,r \times \frac{dt}{T} + A\,b\,u\,u \times \frac{r}{g}\,d\,t$;

whence we get $dt = \frac{T\,du}{2\pi\sqrt{\frac{aa-uu}{rr}}} \times \frac{1}{1 + Abuu \times \frac{T}{2\pi g}}$.

But $\frac{1}{1+Abuu \times \frac{T}{2\pi g}} = 1 - Abuu \times \frac{T}{2\pi g} +$, &c.

where the other terms may be neglected which contain the powers of A, because A is a very small fraction.

We shall have, therefore, $d\,t = \frac{T\,d\,u}{2\,\pi\sqrt{\frac{aa-uu}{rr}}} \times \left(1 - A\,b\,u\,u \times \frac{T}{2\,\pi\,g}\right)$; and the instantaneous precession will be $A\,b \times \frac{T}{2\pi} \times \frac{u\,u\,d\,u}{\sqrt{\frac{aa-uu}{rr}}} \times \left(1 - A\,b\,u\,u \times \frac{T}{2\,\pi\,g}\right)$ whose integral for the first term, in the solsticial point, where $u = \frac{a}{r}$, is $A\,b \times \frac{T}{2\pi} \times \frac{aa\pi}{4rr}$, as was seen in the preceding *Problem*. But the integral of

$u^4\,d\,u$

$\frac{u^4\, du}{\sqrt{\frac{aa}{rr} - uu}} = -u^3 \sqrt{\frac{aa}{rr} - uu} - \frac{3}{4} u \times \left(\frac{aa}{rr} - uu\right)^{\frac{3}{2}}$

$+ \frac{3}{8} \frac{aa}{rr} \int du \sqrt{\frac{aa}{rr} - uu}$, or simply, $= \frac{3}{4} \frac{aa}{rr} \int du$

$\sqrt{\frac{aa}{rr} - uu}$, when we suppose, after the integration,

$\frac{a}{r} = u$; and $= \frac{3}{4} \frac{aa}{rr} \times \frac{a\pi}{2r} = \frac{3}{4} \frac{aa}{rr} \times \frac{a a \pi}{4 r r}$. Therefore the preceſſion from the ſun's departure from the equinox till his arrival at the ſolſtice, is $A b \times \frac{T}{2} \times \frac{a a}{4 r r} \times$

$\left(1 - A b \times \frac{T}{2 \pi g} \times \frac{3}{4} \frac{aa}{rr}\right) = \left(g \text{ being } = \frac{aa}{r}\right) A b \times$

$\frac{T}{2r} \times \frac{g}{4} \times \left(1 - A b \times \frac{T}{2 \pi r} \times \frac{3}{4}\right)$.

When the motion of the node is not very ſenſible, it will ſuffice to make uſe of the *formula* of the preceding *Problem*, which is the firſt term of this: But when that motion is ſenſible, which is the caſe of the moon's node, when it is to be determined for a quarter of the revolution of the ſun, it will be requiſite to employ this *formula*, which contains two terms of the ſeries. Nay, it would be very eaſy to take a greater number of terms, ſo as to neglect nothing; but this would be abſolutely uſeleſs, becauſe of the quick converging of the ſeries.

PRO-

PROBLEM VII.

14. *To find the alteration of the inclination of the earth's axe to the ecliptic, during the sun's passage from the equinox to the solstice, or from the solstice to the next equinox; which is the same, only affected with a contrary sign, as has been seen in* Prob. V.

By *Art.* 8. we have $aPp = AVu \times adt$; and by *Prob.* V. the instantaneous variation of the inclination of the earth's axe to the ecliptic is $AVu \times adt \times \frac{PM}{Pp} = AVu \times adt \times \frac{\overline{\text{♋}D}}{PD}$, because of the similar triangles PpM, $PD\text{♋}$, $=$ (taking the denominations of *Prob.* VI.) $AVu \times a \times \frac{T}{2\pi} \times \frac{r \times d\overline{CD}}{CD \times \overline{\text{♋}D}} \times \frac{\overline{\text{♋}D}}{PD} = AVu \times$

$$\frac{aT}{2\pi} \times \frac{rd\overline{CD}}{CD \times PD} = AVu \times \frac{aT}{2\pi} \times \frac{r \times \frac{du}{uu}}{\frac{ar}{u} \times \frac{a}{u}} = AV \times \frac{T}{2\pi}$$

$\times udu$, whose integral is $\frac{1}{2} Ar \times \frac{T}{2\pi} \times uu$; and when the sun is at the point ♋, we have $1 : u :: r : a$, and $a = ru$; therefore the preceding integral is then $\frac{1}{2} A \times \frac{T}{2\pi r} \times aa$. Which was to be found.

Remark.

15. When the motion of the node is not insensible, you are to take the value of dt of the *Remark* that follows

follows *Art.* 12. which gives $dt = \frac{T}{2\pi} \times \frac{r \times d\overline{CD}}{\overline{\overline{CD}} \times \overline{\overline{SD}}} \times$ $(1 - Ab \times \frac{T}{2\pi g} \times uu)$, and the inſtantaneous variation of the inclination of the earth's axe to the ecliptic will be $Ar \times \frac{T}{2\pi} \times u\,du \times (1 - Ab \times \frac{T}{2\pi g} \times uu)$, whoſe integral, when the ſun arrives at the ſolſtice, where we have $ru = a$, becomes $\frac{1}{2} Aaa \times \frac{T}{2\pi r} - \frac{1}{4} \frac{Aa^4}{r^2} \times \frac{T}{2\pi r} \times Ab \times \frac{T}{2\pi g} =$ (becauſe $g = \frac{aa}{r}$) $\frac{1}{2} Aaa \times \frac{T}{2\pi r} \times (1 - \frac{1}{2} Ab \times \frac{T}{2\pi r})$; and this is the *formula* to be made uſe of when the motion of the nodes is ſenſible.

SECT II.

Of the motion of the pole of the equator of the earth cauſed by the action of the moon.

16. Whatever has been ſaid in *Sect.* I. of the preceſſion, and of the oſcillation of the earth's axe on the plane of the ecliptic, through the action of the ſun, is equally applicable to the action of the moon; it being requiſite only to ſubſtitute every-where the moon inſtead of the ſun; L inſtead of S, and λ inſtead of s; the time which the moon employs in completing her revolution about the earth, inſtead of the time the ſun employs in performing his

his revolution about the earth, or M instead of T. Lastly, the lunar orbit, instead of the ecliptic. Likewise, instead of A, which (*Art.* 8.) was equal to $\frac{3S}{s^3m}$, we must put $B = \frac{3L}{\lambda^3 m} \times \frac{\mathrm{a}a}{1-2\mathrm{a}}$.

17. We shall find then, by *Art.* 12. that the arc described by the pole P of the earth parallel to the plane of the lunar orbit, whilst the moon makes one quarter of a revolution about the earth, is $Bb \times \frac{g}{2r} \times \frac{M}{4}$, which gives the arc described by the mean motion in an instant, dt, equal to $Bb \times \frac{g}{2r} \times dt = Bb \times \frac{aa}{2rr} \times dt$; putting, instead of g, its value $\frac{aa}{r}$.

18. We shall find also, by *Art.* 14. that the variation of the inclination of the earth's axe to the plane of the lunar orbit, when the moon is at its point of station 90 degrees distant from the node of the equator with the lunar orbit, is $\frac{1}{2} B \times \frac{M}{2\pi r} \times aa$, whereby the angle of inclination of the earth's axe to the plane of the lunar orbit is increased when the moon is at its point of station.

19. We therefore should have nothing to add, if the plane of the lunar orbit were the same as that of the ecliptic, or if it always retained the same position: But as this plane continually varies its position, and its pole describes a small circle about the pole of the ecliptic, from whence it is always about 5 degrees distant; it still remains that we examine what is the

preceffion caufed by the moon, with refpect to the ecliptic; which may be done by refolving the motion of the earth's pole, parallel to the plane of the lunar orbit, into two motions, one whereof is parallel to the plane of the ecliptic, and the other perpendicular thereto, and which varies the inclination of the earth's axe to the ecliptic. This is what we fhall determine in the following *Problem.*

Problem VIII.

20. *Having given, the motion of the pole* P *of the earth parallel to the plane of the lunar orbit, to find its motion parallel to the plane of the ecliptic.*

* *C P, fig.* 3. *n*° 1. & *n*° 2. is the femi-axe of the earth. The plane of the figure is the plane of the folfticial colure; *C Z* is the axe of the ecliptic, confequently perpendicular to *C* ♋; ♋ the folfticial point, and *P C* ♋ the folfticial angle: *P Z* is parallel to *C* ♋, and perpendicular to *C Z*; and it is equal to the line *C H* of *fig.* 2. The circumferencc l, l', l^2, l^3, l is the circumference defcribed by the pole, l, of the lunar orbit; L, L', L^2, L^3 is the fection of the cone $C\,l, l'\,l^2\,l^3\,l$ prolonged, by the plane that is tangential at the pole of the earth, *P*.

Let l^0 be any place of the pole of the moon's orbit. If upon the point *P* be erected *P R*, perpendicular to the plane $P\,C\,l^0$, this perpendicular will be the direction of the motion of the earth's pole, parallel to

* The lines which are in the plane which touches the earth's pole l, are feparately reprefented in *fig.* 2. *n*° 3.

 the

the plane of the lunar orbit, the plane $P C l^{\circ}$ being perpendicular to that plane, $P C l^{\circ}$ being the axe thereof.

Letting fall from the point R the perpendicular $R M$ upon the line ♋ $P Z$, the motion of the pole of the earth according to $P R$ may be refolved into two; the one according to $P M$, the other parallel and equal to $R M$; and this latter expreffes the preceffion with regard to the ecliptic, and the former gives the alteration of the inclination of the earth's axe to the plane of the ecliptic.

The motion of the pole reprefented by $M R$, which gives the preceffion with regard to the ecliptic, is always on the fame fide during the entire revolution of the pole of the lunar orbit, and is the fame on either fide the point l at the like diftance from it; the motion according to $P R$ being then the fame, and always on the fame fide, as has been explained in *Prob.* V.

* The motion according to $P M$ is the fame on either fide of the point l at the fame diftance from it, but with a contrary fign, that is, in an oppofite direction. For it is manifeft, that if another point, as l°, be taken on the other fide, and at an equal diftance from the point l, the point M, where the perpendicular $R M$ falls, will be on the other fide with regard to P, than the point M, fince the angle $R P Z'$, which is here obtufe, will then be acute, and lefs than the right angle of an angle equal to $L^{\circ} P Z'$.

* The point l° placed on the other fide of the point l, that is, between l and l^{3}, is not marked in the *fig.* nor the point M' where the perpendicular $K M'$ falls; for the fake of not embarraffing the *fig.* with too many lines, it being eafy to conceive.

For

For this reason, the motion according to *PM* must produce an oscillation of the earth's axis, whose period is the time of an entire revolution of the pole of the lunar orbit about the pole of the ecliptic.

Problem IX.

21. *To find the quantity of the precession caused by the moon with regard to the ecliptic; that is, the arc parallel to the ecliptic, run through by the earth's pole* P, *during the time of a semi-revolution* $l\,l^1\,l^2$ *of the pole* l *of the lunar orbit.*

Let there be drawn the lines *NK*, $l^o K$ perpendicular to the axe of the earth *CP*, and parallel to the lines *NP*, $l^o P$, *fig.* 3. *no* 1. & *n°* 2.

The triangles *PRM*, *PNL°*, *KNl°* are right-angled and similar; for the three sides of the first are perpendicular to the three sides of the second, and the three sides of the second are parallel to the three sides of the third.

Still retaining the foregoing denominations, let $zl = x$; $ZN = z$; $PZ = g$, as in *fig.* 2. the line *CH*; $CZ = f$; Cl or $Cl^o = C$.

We shall have $Nl^o = \sqrt{xx - zz}$; $NP = g - z$; $NK = \frac{f}{a} \times \overline{g - z}$, or $\frac{f}{a} \times \overline{g + z}$, when the point l^o lies on the other side of l^1; $PR = \frac{g}{a} \times \overline{g - z}$, or $\frac{g}{a} \times \overline{g + z}$ when the point l^o is on the other side of l^1. The triangle Cl^oK is here similar to the triangle whose three sides are represented by the letters *a*, *b*, *r*, since

since $\frac{b}{r}$ expresses here the sine of the angle of inclination of the earth's axe to the plane of the lunar orbit, and $\frac{a}{r}$ is its cosine; we shall have then $l^{\circ} K = \frac{ac}{r}$; $CK = \frac{bc}{r} = CP - PK = a - \frac{g}{a} \times \overline{g-z} = \frac{aa-gg+gz}{a} = \frac{ff+gz}{a}$, or $\frac{ff-gz}{a}$, when the point l° is on the other side with respect to l^{I}; $\frac{RM}{PR} = \frac{NK}{l^{\circ}K} = \frac{\frac{f}{a} \times \overline{g-z}}{\frac{ac}{r}}$, or $\frac{\frac{f}{a} \times \overline{g+z}}{\frac{ac}{r}}$ for the other side of l^{I}; $\frac{\overline{PM}}{\overline{PR}} = \frac{Nl^{\circ}}{l^{\circ}K} = \frac{\sqrt{xx-zz}}{\frac{ac}{r}}$: Lastly, the difference of the arc $l^{\text{I}} l^{\circ} = du = \frac{x\,dz}{\sqrt{xx-zz}} = \frac{2\pi x}{nT} \times dt$, calling nT the time which the pole of the lunar orbit employs in performing its revolution in the circumference $2\pi x$, whence we get $dt = \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx-zz}}$.

The space run through in an instant dt by the pole P of the earth parallel to the plane of the lunar orbit being

being by *Art.* 17. $= Bb \times \frac{aa}{2r} \times dt$; and by *Prob.* VIII. the ſpace run through parallel to the plane of the ecliptic, will be $Bb \times \frac{aa}{2rr} \times dt \times \frac{xI}{Px} =$ (by the preceding denominations) $Bb \times \frac{aa}{2rr} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}}$

$$\times \frac{\frac{f}{a} \times \overline{g-z}}{\frac{ac}{r}} = B \times \frac{b}{r} \times \frac{nT}{4\pi} \times \frac{dz}{\sqrt{xx-zz}} \times \frac{f \times \overline{g-z}}{c};$$

and putting inſtead of $\frac{b}{r}$, its value, which was found to be $\frac{ff+gz}{ac}$, we ſhall have $B \times \frac{nT}{4\pi} \times \frac{f}{acc} \times \overline{ff+gz} \times \overline{g-z} \times \frac{dz}{\sqrt{xx-zz}}$; and when the point l° is on the other ſide with regard to l^{I}, we ſhall have $B \times \frac{nT}{4\pi} \times \frac{f}{acc} \times \overline{ff-gz} \times \overline{g+z} \times \frac{dz}{\sqrt{xx-zz}}$. Therefore always taking the preceſſion for the two points together placed on either ſide of l^{I}, and at an equal diſtance from it, we ſhall have, by joining the two values now found, after making reduction, $B \times \frac{nT}{4\pi} \times \frac{f}{aa} \times \overline{2gff - 2gzz} \times \frac{dz}{\sqrt{xx-zz}}$, or $B \times \frac{nT}{2\pi} \times \frac{fg}{aa} \times \overline{ff-zz} \times \frac{dz}{\sqrt{xx-zz}}$.

But

But the integral of $ff \times \frac{dz}{\sqrt{xx - zz}} = \frac{ff}{x} \times \frac{dz}{\sqrt{xx - zz}}$ is $\frac{ff}{x} \times \frac{x\pi}{2} = \frac{ff\pi}{2}$, when we take all the points l°, from l^{1} to l; and the integral of $\frac{zzdz}{\sqrt{xx - zz}}$, which is $- z\sqrt{xx - zz} + \int dz \sqrt{xx - zz}$; when, after the integration, we make $z = x$, becomes $\int dz\sqrt{xx - zz}$, or the area of the quadrant $l^{1} z l$, which is $= \frac{x\pi}{2} \times \frac{x}{2} = \frac{xx\pi}{4}$.

Therefore the integral of $B \times \frac{nT}{2\pi} \times \frac{fg}{acc} \times \overline{ff - zz} \times \frac{dz}{\sqrt{xx - zz}}$ is $= B \times \frac{nT}{2\pi} \times \frac{fg}{acc} \times \left(\frac{ff\pi}{2} - \frac{xx\pi}{4}\right)$ $= B \times \frac{nT}{2} \times \frac{fg}{acc} \times \left(\frac{ff}{2} - \frac{xx}{4}\right) = B \times \frac{nT}{2} \times \frac{gf^{3}}{2acc} \times \left(1 - \frac{xx}{2ff}\right)$.

This is the ſum of all the inſtantaneous preceſſions for all places of the lunar orbit from l to l^{2}, becauſe we have always taken two points together placed on either ſide of l^{1}.

And this is the quantity of the preceſſion with regard to the ecliptic, during the time of a ſemi-revolution of the lunar orbit. Which was to be found.

PRO-

PROBLEM X.

22. *To find the quantity of the nutation, or of the variation of the inclination of the earth's axe to the ecliptic, caused by the moon during the semi-revolution* l l' l'' *of the pole of the lunar orbit.*

The space run through in an instant dt by the pole of the earth, parallel to the plane of the lunar orbit, being, by *Art.* 17. $= Bb \times \frac{aa}{2rr} \times dt$, the instantaneous nutation for any one place l° of the lunar orbit will be, by *Prob.* VIII. $= Bb \times \frac{aa}{2rr} \times dt \times \frac{PM}{PR}$ $=$ (by the denomination of the preceding *Problem*) $Bb \times \frac{aa}{2rr} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}} \times \frac{\sqrt{xx - zz}}{\frac{ac}{r}} = B \times \frac{b}{r}$ $\times \frac{aa}{2} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}}$. Putting instead of $\frac{b}{r}$ its value, which was found to be $\frac{ff + gz}{ac}$, we shall have $B \times \frac{nT}{2\pi \times 2cc} \times \sqrt{ff + gz} \times dz$; and when the point l° is on the other side with regard to l', we shall have $B \times \frac{nT}{2\pi \times 2cc} \times \sqrt{ff - gz} \times dz$.

Taking therefore the nutation for the two points together, l°, placed on either side of l', we have $B \times$

$B \times \frac{nT}{2\pi \times 2cc} \times 2ffdz$, whose integral, when z is $= x$, is $B \times \frac{nT}{2\pi \times 2cc} \times 2ffx = B \times \frac{nT}{2\pi} \times \frac{ff}{cc} \times x$.

This is the quantity of the nutation caused by the moon during the semi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit. Which was to be found.

COROLLARY.

23. By *Prob.* IX. the quantity of the precession caused by the moon, with regard to the ecliptic, is, during the space of a semi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit, $B \times \frac{nT}{2} \times \frac{gf^3}{2acc} \times \left(1 - \frac{1}{2} \times \frac{xx}{ff}\right)$.

By the preceding *Problem*, the quantity of the nutation during the same time of a semi-revolution of the pole of the lunar orbit $l\,l^1\,l^2$, caused also by the moon, is $B \times \frac{nT}{2\pi} \times \frac{ff}{cc} \times x$.

Therefore the precession, during the time of a semi-revolution of the pole of the lunar orbit, is to the nutation, during the same time,

As $\frac{gf}{2a} \times \left(1 - \frac{1}{2}\frac{xx}{ff}\right)$, is to $\frac{x}{\pi}$; or as $gf\pi \times \left(1 - \frac{1}{2}\frac{xx}{ff}\right)$ is to $2ax$.

But the precession is performed in a parallel to the ecliptic, whereof g is the radius; and the nutation is performed in the plane of the solsticial colure, and in a circle whereof a is the radius: Di-

 viding,

viding, therefore, each circular arc by its radius, we shall have the quantity of the angle: And

The angle of the preceſſion, during the time of a ſemi-revolution of $l\,l^1\,l^2$ of the pole of the lunar orbit, will be to the angle of the nutation, in the ſame time,

$$\text{As } f\,\pi \times \left(1 - \tfrac{1}{2}\,\frac{xx}{ff}\right) \text{ is to } 2\,x\text{, or as } \pi\left(1 - \tfrac{1}{2}\,\frac{xx}{ff}\right)$$

is to $\frac{2x}{f}$.

SCHOLIUM.

24. Whatever has been ſaid as to the moon and the ſun, may be equally applicable to all the other planets; but as the effect of the action of the reſt of the planets upon the earth is not ſenſible, we ſhall not take it into conſideration.

SECT. III.

Of the motion of the pole of a ring, cauſed by the action of the ſun: or of the motion of the pole of the moon's orbit.

25. We have ſeen, in *Art.* 9. that if, inſtead of the terreſtrial ſpheroid, we were to ſuppoſe a ſimple ring placed at the circumference of the equator, the motion of the pole of ſuch ring would be found by the ſame *formula* as that for the motion of the pole of the earth; ſuppoſing only, in that *formula*, $\frac{a}{1-2a} = 1$.

By

By this means we may apply all the *formulæ* which have been found in *Sect.* I. to the motion of the nodes of this ring, and to the alteration of the inclination of its axe to the plane of the ecliptic.

But by the *Remark* on *Art.* 6. the motion of the plane of this ring is the ſame, whether the ring be entire, or there be only a ſingle point which circulates in the ring's circumference: Whence it follows, that

Theſe ſame *formulæ* do likewiſe give the motion of the pole of the orbit of a moon or ſatellite; the motion of the node of the orbit of ſuch moon in the ecliptic; and the variation of the inclination of its axe of the orbit to the plane of the ecliptic: Obſerving to put the time of the revolution of this moon about the ſun, inſtead of the time of the earth's revolution about the ſun, and the motion of the moon in its orbit inſtead of the motion of a point of the equator.

Remark.

It may be obſerved, that although the motion of the pole of a ring be the ſame as that of a moon, during the time of the revolution of the ring, or of the moon, which is the ſame; yet there are ſome particular motions which take place when there is only one moon revolving in the circumference of the ring, and which ceaſe to exiſt when it is an entire ring that revolves.

For example; in *Art.* 2. the force, according to $L\gamma$, *fig.* 1. *n°* 2. of the point L, is deſtroyed by an equal and directly oppoſite force of another point λ placed on the other ſide, and at the ſame diſtance from the point E, as the point L, when the ring is entire;

 but

but when there is but one point, as L, or, which is the ſame, one moon, which revolves in the circumference of the ring; this motion ſubſiſts, and diſturbs the motion of the moon in the plane of its orbit.

In like manner the force of the point L parallel to γC, or according to $c' L$, is deſtroyed, in a ring, by the force of another equal point λ' placed on the other ſide with regard to ☊, and at the ſame diſtance as the point L. But this force ſubſiſts in the caſe where there is but one ſingle point L, or one moon, and this motion diſorders, beſides, the motion of the moon in the plane of its orbit.

Laſtly, If the ring, which circulates about the centre C, be elliptical, the motion of every point of the ring in the orbit is proportional to its diſtance from the centre; becauſe, on account of the continuity of the ring, all the points complete their revolution in the ſame time, and have equal angular motions; but when it is but one ſingle moon that revolves, in an elliptic orbit, it deſcribes areas proportional to the times, and has an angular motion ſo much the greater, as it approaches nearer the centre; which is the contrary of the motion of the points of the elliptic ring, the more diſtant whereof from the centre, have the moſt motion. And it ariſes from hence, that the motion of the pole of the lunar orbit is much the ſame as that of the ring, taking it during the entire revolution; but that there is a ſmall variation, or inequality, in the inſtantaneous motion of the pole of a moon; whereas this inſtantaneous motion is conſtant through every inſtant of the revolution of the ring.

I ſhall

I ſhall ſpend no more time on the ring, with reſpect to thoſe motions which have nothing to do with the ſubject in queſtion; but they ſhall be thoroughly diſcuſſed, and applied, in a ſecond memoir, which I ſhall publiſh, on the particular motions of the planets.

SECT. IV.

Containing the application of the formulæ *of the preceding Sections.*

26. The motion of the ſun, or, to ſpeak more exactly, of the earth, in the ecliptic, is $\frac{2\pi s}{T}$, which is the ſpace divided by the time.

27. The earth's central force will therefore be $\left(\frac{2\pi s}{T}\right)^2 \times \frac{1}{2s}$, the central force being equal to the ſquare of the motion divided by the diameter of the circumference in which the body moves; and the earth's central force being the force $\frac{S}{ss}$ wherewith it is attracted towards the ſun; we ſhall have $\frac{S}{ss} = \left(\frac{2\pi s}{T}\right)^2 \times \frac{1}{2s} = \frac{2\pi\pi s}{TT}$; whence we get $\frac{S}{s^3} = \frac{2\pi\pi}{TT}$.

28. The motion of a point of the equator whoſe radius is a, will be, calling the time of its revolution t, $= \frac{2A\pi}{t}$, which is the ſpace divided by the time. We have therefore $m = \frac{2A\pi}{t}$.

29. By *Art.* 8. $A = \frac{3S}{s^3 m} \times \frac{\text{a}a}{1-2\text{a}} =$ (by *Articles* 27. & 28.) $3 \times \frac{2\pi\pi}{TT \times 2\text{a}\pi} \times \frac{\text{a}a}{1-2\text{a}} = \frac{3t}{TT} \times \pi \times \frac{\text{a}}{1-2\text{a}}$ $=$ (by *Art.* 31. following) $\frac{3t}{TT} \times \pi \times \frac{1}{176}$.

30. Putting q for the ratio of the lunar force $\frac{L}{\lambda^3}$, to the solar force $\frac{S}{s^3}$, we shall have $\frac{L}{\lambda^3} = \frac{S}{s^3} \times q$; and B, which (by *Art.* 16.) is $= \frac{3L}{\lambda^3 m} \times \frac{\text{a}a}{1-2\text{a}}$ will also be $= \frac{3S}{s^3 m} \times q \times \frac{\text{a}a}{1-2\text{a}} =$ (by *Art.* 29.) $\frac{3qt}{TT} \times \pi \times \frac{1}{176}$.

31. Supposing $\text{a} = \frac{1}{178}$, which is conformable to the observations of the gentlemen of the *Parisian* academy, we shall have $\frac{\text{a}}{1-2\text{a}} = \frac{1}{178 \times 1 - \frac{2}{178}} = \frac{1}{176}$.

PROBLEM XI.

32. *To find the precession caused by the sun during the time between the equinox and the solstice; that is to say, in a quarter of a year, or three months.*

By *Art.* 12. the precession, during this time, is $Ab \times \frac{T}{2} \times \frac{g}{4r}$. Putting, instead of A, the value found

found in *Art.* 29. we shall have $\frac{3t}{TT} \times \pi \times \frac{1}{176} \times \frac{bT}{2}$ $\times \frac{g}{4r} = \frac{3t}{T} \times \frac{1}{176} \times \frac{b}{r} \times \frac{\pi g}{8}$.

But $\frac{b}{r}$ is the ratio of the sine of the inclination of the axe of the earth to the ecliptic, to the radius, or of the sine of 66 degrees 32 minutes to the radius, $= \frac{9172919}{10000600}$; and πg is the semi-circumference of the parallel to the ecliptic wherein the pole of the earth moves, and, consequently, equal to 180 deg.

t, which is the time of the revolution of a point of the equator, is equal to one day; and T equal to $365\frac{1}{4}$ days; we shall have, then, by substituting all the values in the preceding *formulæ*, $\frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{1}{176} \times \frac{180}{8}$ degrees. This is the precession produced by the sun during the space of 3 months.

33. Multiplying by 4, we shall have the annual precession caused by the sun $= \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{1}{176} \times \frac{180}{2}$ degrees, $= \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000} \times \frac{45}{88}$ degrees, $=$ (after due reduction) $13''. 52'''. 11''''$.

PRO-

Problem XII.

34. *To find the preceſſion cauſed by the moon, during the time of a ſemi-revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about* $9\frac{1}{3}$ *years, or* $9\frac{1}{3} \times$ T, *as* T $=$ 1 *year.*

By *Art.* 21. we have found this preceſſion $= B \times \frac{nT}{2} \times \frac{gf^3}{2acc} \times (1 - \frac{1}{2} \times \frac{\varkappa\varkappa}{ff})$.

By *Art.* 30. $B = \frac{3qt}{TT} \times \pi \times \frac{1}{176}$: Subſtituting this value in the preceding *formula*, it becomes $\frac{3qt}{T} \times \frac{n}{2} \times \frac{f}{2a} \times \frac{ff}{cc} \times \frac{1}{176} \pi g \times (1 - \frac{1}{2} \frac{\varkappa\varkappa}{ff})$.

But $\frac{f}{a}$ is the ratio of the ſine of 66 deg. 32 min. to the radius, $= \frac{9172919}{10000000}$.

$\frac{f}{c}$ is the ratio of the ſine of 85 deg. to the radius, the angle $l\,c\,z$, whereby the pole of the lunar orbit is diſtant from the pole of the ecliptic, being, at all times, about 5 degrees. Therefore $\frac{f}{c} = \frac{9961947}{10000000}$.

$\frac{\varkappa}{f}$ is the ratio of the ſine of 5 deg. to the ſine of 85 degrees, $= \frac{871557}{9961947}$.

 Therefore

Therefore substituting these values, and those of t, T, and πg found in the preceding *Problem*, we shall find that the precession, during a semi-revolution of the lunar pole, which is $\frac{nT}{2}$, by *Art.* 21. becomes

$$= q \times \frac{n}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2} \times \left(\frac{9961947}{10000000}\right)^2 \times \frac{1}{176}$$

$$\times\ 180 \text{ deg.} \times \left(1 - \frac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right).$$

This is the precession caused by the moon during the time of a semi-revolution of the pole of the lunar orbit, $\frac{nT}{2} = 9\frac{1}{3} \times T$.

35. Therefore the mean precession caused by the moon during the time T, which is the mean annual precession caused by the moon, will be, by dividing the preceding *formula* by $\frac{n}{2}$, $= q \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000200}\right)^2 \times \frac{180 \text{deg.}}{176} \times \left(1 - \frac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right)$$

where it only remains to determine the value of q, which, by *Art.* 30. is the ratio of the lunar force to the solar force; and making this ratio $= \frac{5}{2}$, conformable to the observations of the tides related by Mr. *Daniel Bernoulli*, we shall have for the mean annual precession caused by the moon, $\frac{5}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000000}\right)^2 \times \frac{180 \text{deg.}}{176} \times \left(1 - \frac{1}{2} \times \left(\frac{871557}{9961947}\right)^2\right)$$

$=$ (after due reduction) $34''.\ 16'''.\ 51''''$.

Problem XIII.

36. *To find the nutation of the earth's axe caused by the moon, during the time of a semi-revolution of the pole of the moon's orbit, that is, in $9\frac{1}{3}$ years.*

By *Art.* 23. $\pi \times \left(1 - \frac{1}{4} \times \frac{xx}{ff}\right)$ is to $\frac{2x}{f}$, as the angle of the preceſſion, found in the preceding *Problem* for the time $\frac{nT}{2}$ of the ſemi-revolution of the pole of the lunar orbit, is to the nutation during the ſame time. By it, therefore, may be found the nutation, having the quantity of the preceſſion of the preceding *Prob.* or, reciprocally, knowing the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, the preceſſion will be known to a geometrical certainty, for the ſame time. The ratio, in numbers, is $\frac{1095798}{61236}$.

37. But we may alſo know the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, by the *formula* of *Art.* 22. which is $B \times \frac{nT}{2} \times \frac{ff}{cc} \times x$; which, without alteration, may be multiplied by $\frac{a\pi}{a\pi}$, or by $\frac{a\pi}{a \times \frac{22}{7}}$; ſince π, which is the ratio of the circumference to the diameter, is $\frac{22}{7}$, and we ſhall have

have the *formula* $B \times \frac{nT}{2} \times \frac{ff}{cc} \times \varkappa \times \frac{a\,\pi}{a \times \frac{22}{7}} = B \times \frac{nT}{2}$ $\times \frac{f}{a} \times \frac{f}{c} \times \frac{\varkappa}{c} \times \frac{7}{22} \times a\,\pi$; which, by substituting the values of the preceding *Problem*, and 180 degrees instead of $a\,\pi$, which is the semi-circumference of the circle wherein the nutation is performed, becomes $\frac{3qt}{T} \times \frac{n}{2} \times \frac{9172919}{10000000} \times \frac{9961947}{10000000} \times \frac{871557}{10000000} \times \frac{7}{22} \times \frac{180}{176}$ deg. $= \frac{3}{365\frac{1}{4}} \times \frac{5}{2} \times 9\frac{1}{3} \times \frac{9172919}{10000000} \times \frac{9961947}{10000000} \times \frac{871557}{10000000} \times \frac{7}{22} \times \frac{180}{176}$ deg. equal, after reduction, to $17''.\ 51'''.\ 14''''$.

Problem XIV.

38. *To find the difference of the inclination of the earth's axe, between the point of the equinox and that of the solstice, caused by the action of the sun.*

By *Art.* 14. the *formula* is $\frac{1}{2} A \times \frac{T}{2\pi r} \times a\,a$.

Putting, instead of A, its value of *Art.* 29. instead of $\frac{a}{r}$, which is the ratio of the sine of the obliquity of the ecliptic to the radius $\frac{3982155}{10000000}$, we shall have $\frac{1}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{1}{2 \times 176} \times \frac{3982155}{10000000} \times a$; and putting,

 in-

instead of a, $\frac{a\pi}{\pi} = \frac{a\pi}{\frac{22}{7}} = \frac{7}{22}\, a\,\pi = \frac{7}{22} \times 180$ deg.
(since $a\,\pi$ is the semi-circumference of the circle wherein this inclination is performed) we shall have $\frac{3}{365\frac{1}{4}} \times \frac{1}{4 \times 176} \times \frac{3982155}{10000000} \times \frac{7}{22} \times 180$ deg. $=$ (after reduction) $57'''.\ 25''''$.

Problem XV.

39. *To find the difference of the inclination of the earth's axe to the plane of the lunar orbit, when the moon arrives at its point of station, 90 degrees distant from the node of the equator with the lunar orbit, this difference of inclination being caused by the moon.*

By *Art.* 18. the variation of inclination will be $\frac{1}{2} B \times \frac{M}{2\pi r} \times a\, a$.

Putting, instead of B, its value of *Art.* 30.; in-instead of M, which expresses the time of the moon's revolution about the earth, $27\frac{1}{3}$ days; instead of $\frac{a}{r}$, the mean value $\frac{3982155}{10000000}$; I say *mean value*, because the earth's axe is not always equally inclined to the plane of the lunar orbit, as has been shewn in *Art.* 19. *&c.* we shall have $\frac{1}{2} \times \frac{3 \times \frac{5}{2}}{(365\frac{1}{4})^2} \times \frac{27\frac{1}{3}}{2 \times 176} \times \frac{3982155}{10000000} \times a =$ (putting, as in the pre-

ceding

ceding *Problem*, instead of *a*, $\frac{7}{22} \times 180$ degrees) $\frac{1}{2} \times$

$\frac{3\frac{5}{7}}{(365\frac{1}{4})^2} \times \frac{27\frac{1}{3}}{2 \times 176} \times \frac{3982155}{10000000} \times \frac{7}{22} \times 180$ degrees, $=$ (after reduction) $10'''. 19''''$.

Problem XVI.

40. *To find the motion of the node of* Jupiter's *fourth satellite, caused by the sun, during the time of* $\frac{1}{4}$ *of a revolution of that satellite about the sun.*

We need only make use of the *formula* of *Problem* XI. wherein, conformably to *Art.* 25. we must make $\frac{a}{1-2a} = 1$, $t = 16\frac{3}{4}$ days, being the time of the revolution of the satellite about *Jupiter*; $T = 365\frac{1}{4} \times 12$ days, which is the time of the satellite's revolution about the sun, or about 12 years; $\frac{b}{r}, = 1$, because the orbit of the satellite is but little inclined to the ecliptic, and, consequently, the axe thereof nearly at right angles thereto.

We shall have then $\frac{3t}{T} \times \frac{\pi g}{8}$, or $\frac{3 \times 16\frac{3}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

This is the motion of the node of this satellite in 3 years, or during $\frac{1}{4}$ of its revolution about the sun.

Remark.

41. By this may be found the motion of the nodes of the rest of *Jupiter*'s satellites. For it appears, from the *formula*, that the motion of the nodes of different

different ſatellites of the ſame planet, is as the times of their revolutions about the planet.

Thus the motion of the nodes of the 3d ſatellite, in the ſpace of 3 years, is $\frac{3 \times 7\frac{1}{6}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 2d is $\frac{3 \times 3\frac{1}{2}\frac{1}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 1ſt is $\frac{3 \times 1\frac{3}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

PROBLEM XVII.

42. *To find the motion of the nodes of the 5th ſatellite of* Saturn, *cauſed by the ſun, during the time of* $\frac{1}{4}$ *of the revolution of this ſatellite about the ſun.*

Conformably to the preceding *Problem* we ſhall have, by putting inſtead of t, $79\frac{1}{3}$ days; inſtead of T $365\frac{1}{4} \times 30$ days, which is nearly the time of the revolution of this ſatellite about the ſun, and we ſhall have $\frac{3 \times 79\frac{1}{3}}{365\frac{1}{4} \times 30} \times \frac{180 \text{ deg.}}{8}$.

43. In the ſame manner may be found the motion of the nodes of all the reſt of *Saturn*'s ſatellites.

PROBLEM XVIII.

44. *To find the motion of the moon's node during the time that the sun is passing from the node of the moon's orbit with the ecliptic, to its point of station; that is, in three months, wanting a few days.*

Here we are to make use of the *formula* of *Art.* 13. because the motion of the moon's nodes is very sensible in respect of the motion of the sun.

We shall have $Ab \times \frac{T}{2r} \times \frac{g}{4} \times (1 - Ab \times \frac{T}{2\pi r} \times \frac{1}{4})$; and employing the values of *Prob.* XI. except that here $t = 27\frac{1}{3}$ days, which is the time of the moon's revolution about the earth, and that $\frac{b}{r} =$ to the sine of 85 deg. divided by the radius; the axe of the lunar orbit being inclined in an angle of about 85 degrees to the ecliptic. Therefore $\frac{b}{r} = \frac{9961947}{10000000}$. Lastly, by *Art.* 25. we must take $\frac{a}{1-2a} = 1$, and the preceding *formula* will be changed into this, $\frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9661947}{10000000} \times \frac{180 \text{ deg.}}{8} \times \left(1 - \frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9961947}{10000000} \times \frac{3}{2 \times 8}\right) =$ (after reduction) $4°. 36'. 33''$.

This is the motion of the moon's node during the time that the sun is passing from the node of the moon's orbit with the ecliptic to its point of station; that is, during the time the sun is moving 90 — 4 degrees

grees, 36 minutes, 33 seconds, or $85°. 23'. 27''$. And if the motion of the node be required for one year, 'tis only multiplying the motion, just found, by $\frac{360°}{85°. 23'. 27''.}$, and it will be found $19°. 25'. 39''$. being a very small matter more than the truth, for reasons which will be explained elsewhere, and because we have taken $27\frac{1}{3}$ days for the time of the moon's revolution about the earth, which is really a little less than that quantity.

PROBLEM XIX.

45. *To find the greatest difference of the inclination of the lunar orbit to the ecliptic; which happens when the sun arrives at his point of station with regard to the moon.*

Here the *formula* of *Art.* 15. must be made use of, which gives $\frac{1}{2}Aaa \times \frac{T}{2\pi r} \times \left(1 - \frac{1}{2}Ab \times \frac{T}{2\pi r}\right)$, or, which is the same, $\frac{1}{2}A \times \frac{aT}{2\pi r} \times \frac{a\pi}{\frac{22}{7}} \times \left(1 - \frac{1}{2}A \times \frac{bT}{2\pi r}\right)$, substituting the values as in the preceding *Problem*, and instead of $\frac{a}{r}$, putting $\frac{871557}{10000000}$, which is the ratio of the sine of $5°$ to radius, we shall have $\frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{1}{2 \times 2} \times \frac{871557}{10000000} \times \frac{7}{22} \times 180 \text{ deg.} \times \left(1 - \frac{1}{4} \times \frac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \frac{9961947}{10000000}\right) = 15'. 51''$.

Remark.

Remark.

46. It ſhould be obſerved, that, to ſhorten the computations, I have contented myſelf with taking the times of the revolutions pretty near the truth; but if the utmoſt exactneſs be required, the accurate times of the revolutions muſt be employed.

47. This might be a proper place to add the method of determining the perturbation of the orbit of any planet, as derived from another planet; but ſince this depends upon no other than the very ſame principles that have been made uſe of in this memoir, and as their application will be ſhewn, in its full extent, in the memoir which I am going to print, and intend myſelf the honour of ſending to the *Royal Society*, I ſhall deſiſt, that I may not run this paper to a greater length.

LIX. *A Letter to the Right Honourable* George *Earl of* Macclesfield, *P. R. S. concerning the ages of* Homer *and* Heſiod. *By* George Coſtard, *M. A.*

My Lord,

Read Dec. 13, 1753.

IT ſeems to be an opinion pretty generally received, that Homer and Heſiod lived much about the ſame time. If this be true, and they did ſo, whatever arguments prove the age of one, will equally ſerve for fixing that of the other. What that age was, is indeed not at all agreed on among writers; the only thing in which they con-

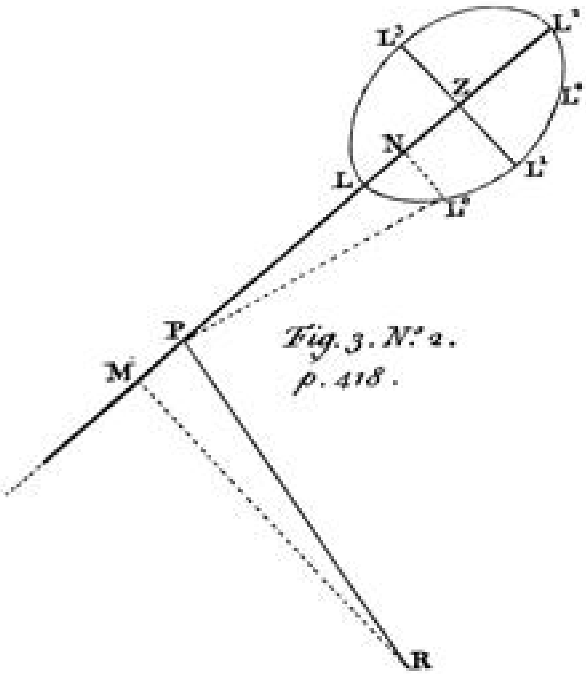


Fig. 3. N.º 2.
p. 418.

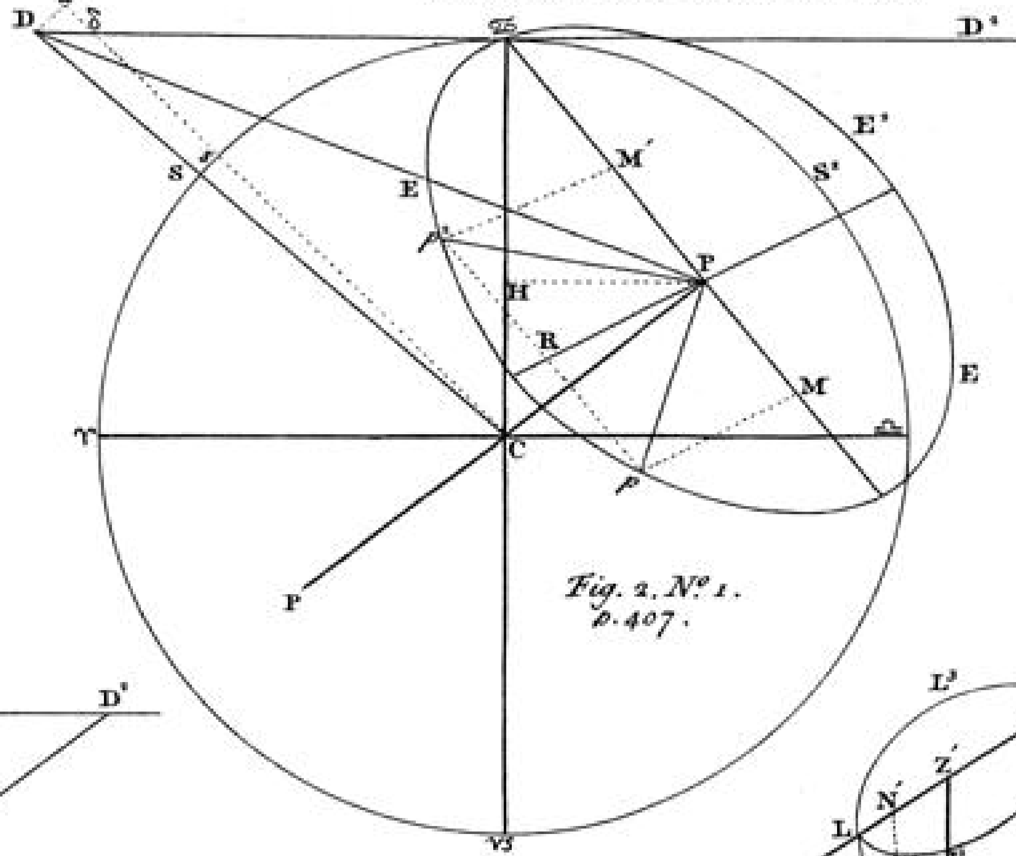


Fig. 2, N.º 1.
p. 407.

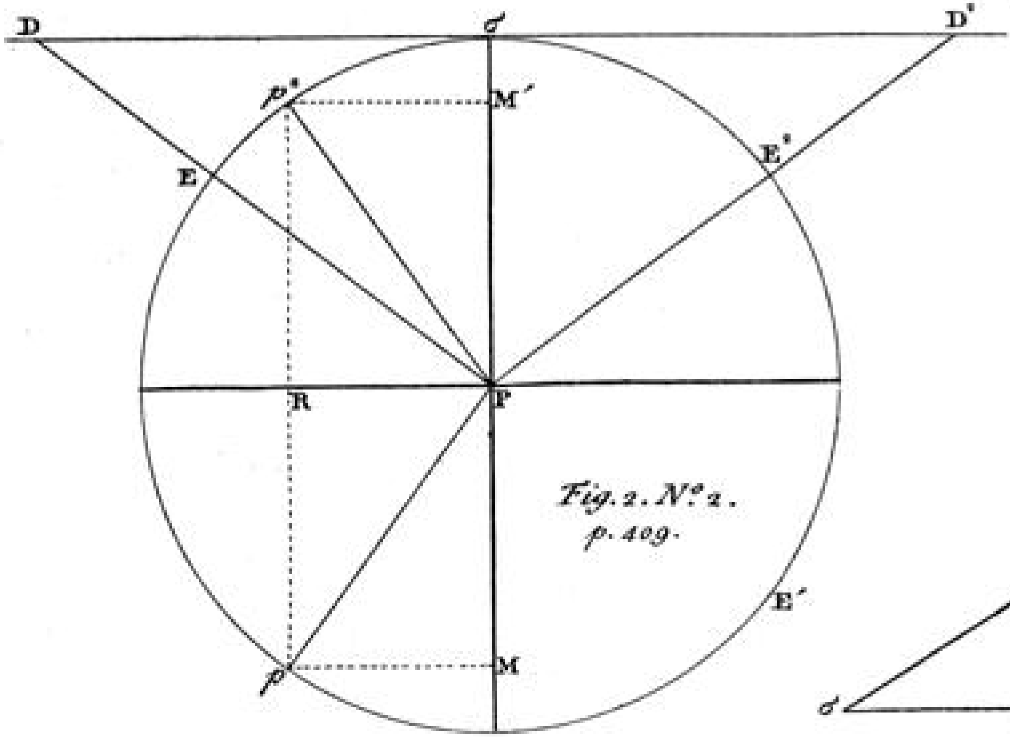


Fig. 2. N.º 2.
p. 409.

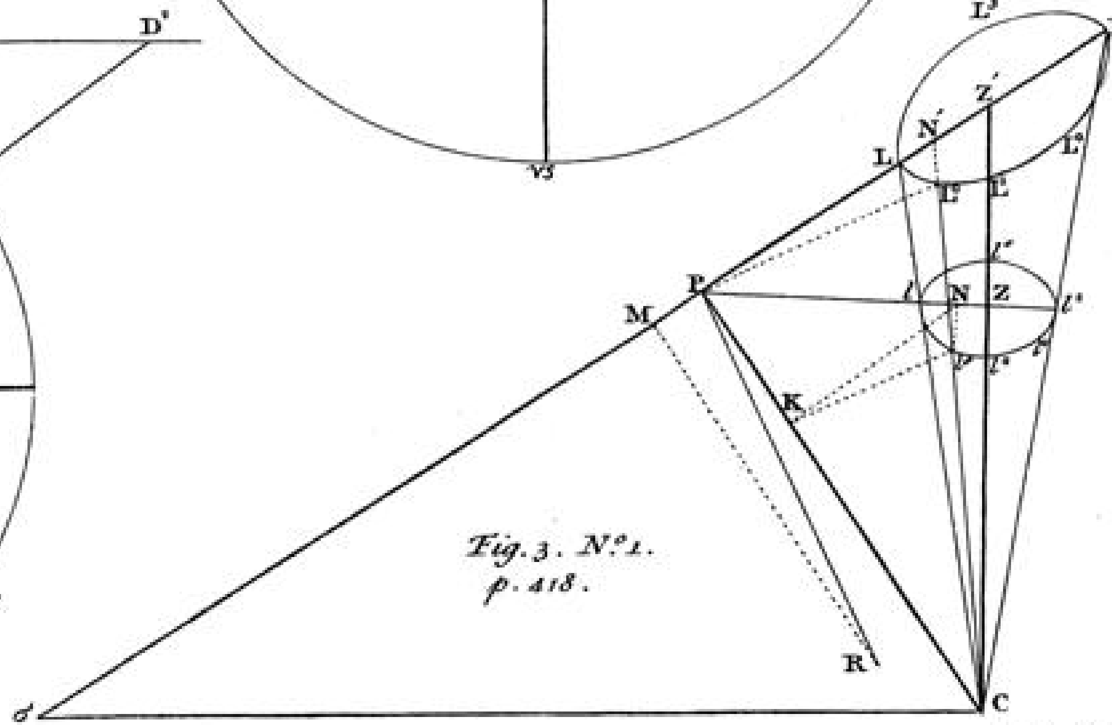


Fig. 3. N.º 1.
p. 418.

J. Mynde sculp.